COHOMOLOGY OF PERFECT LIE ALGEBRAS AND PARTITIONS

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For fixed non-negative integers j and k, denote by $p(j,k,n)$ the number of partitions of n into at most k parts, having largest part at most j. We are interested in explicit formulas of certain one-parameter families of partition numbers, arising in the study of the Satake transform in the context of the Langland's program (the "Beyond Endoscopy Proposal" for GL_2), and also arising in the computation of the adjoint Lie algebra cohomology of certain families of Lie algebras.

We will give a short introduction to Lie algebras and their cohomology by elementary means, and then compute the adjoint cohomology spaces $H^n(\mathfrak{g},\mathfrak{g})$ for the family of perfect, but not semisimple Lie algebras $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V_n$, where V_n denotes the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n+1$. This family is of interest in the context of a conjecture by Pirashvili, which says that a perfect Lie algebra is semisimple if and only if all the adjoint cohomology vanishes.

The computation of the spaces $H^n(\mathfrak{g},\mathfrak{g})$ for $n \geq 0$ involves the Hochschild-Serre spectral sequence and the "explicit plethysm" formula for the multiplicities $N(j,k,n)$ of the irreducible representations V_{jk-2n} arising as a direct summand in the decompostion of $\Lambda^j(V_{j+k-1})$ into irreducible modules. It turns out that $N(j,k,n) = p(j,k,n) - p(j,k,n-1)$ is a difference of the partition numbers from above. We are interested in explicit formulas for these numbers, in the cases where $j \leq 5$ is fixed, and $n = \frac{jk-2}{2}$, or $n = \frac{jk-j+k+1}{2}$. We present some results, such as $N(4,k,2k-1) = 0$ for all $k \geq 1$, or

$$N(4,4\ell-3,6\ell-6) = \left\lfloor \frac{5\ell^2-\ell+16}{12} \right\rfloor + \left\lfloor \frac{5\ell-5}{12} \right\rfloor - \left\lfloor \frac{5\ell-4}{12} \right\rfloor + \left\lfloor \frac{5\ell-3}{12} \right\rfloor .$$

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