The Hadwiger Theorem

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In 1952, Hugo Hadwiger established a remarkable classification result. He proved that all rigid motion-invariant, continuous valuations on the space of convex bodies (compact convex sets) in $\mathbb{R}^n$ are linear combinations of intrinsic volumes. Here, a functional Z defined on convex bodies is called a valuation (or additive) if

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all convex bodies K and L such that $K \cup L$ is again a convex body. In $\mathbb{R}^3$, the Hadwiger theorem states that for every rigid motion-invariant, continuous valuation Z, there are $c_0, c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\mathrm{Z}(K) = c_0 V_0(K) + c_1 V_1(K) + c_2 V_2(K) + c_3 V_3(K)$$

for every convex body in $\mathbb{R}^3$. Here, $V_0(K) = 1$ (the Euler characteristic) and $V_3(K)$ is the 3-dimensional volume of K, while $V_2(K)$ is (up to a constant multiple) the perimeter of K and $V_1(K)$ its mean width.

We will discuss this result, some of its consequences and applications, and some of the many results it inspired. In particular, we will describe a recent functional version of the Hadwiger theorem (joint work with Andrea Colesanti and Fabian Mussnig).