CRITICAL INTERMITTENCY IN RANDOM INTERVAL MAPS

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Abstract. ...

1. Introduction

Intermittency is a type of behaviour observed in certain dynamical systems. These systems alternate between long periods of exhibiting one out of several types of dynamical characteristics (?). In their seminal paper [31] Manneville and Pomeau investigated this phenomenon in the context of transitions to turbulence in convective fluids, see [31, 26, 12], and distinguished several different types of intermittency. An illustrious example of a one-dimensional map with intermittent behaviour is the Manneville-Pomeau map $T : [0,1] \to [0,1]$, $x \mapsto x + x^{1+A}$ for some $A > 0$. Here the intermittent behaviour is caused by the presence of a neutral fixed point. The dynamical properties of this map, the closely related LSV-map [25] and other maps with a single neutral fixed point have been extensively studied over the past decades. It is known for example that such maps admit an absolutely continuous invariant measure and that their statistical properties are determined by the characteristics of the fixed point. See e.g. [29, 34, 18, 19, 11, 10, 16] for results on the Manneville-Pomeau and LSV maps and e.g [32, 14, 22, 30, 35] for other related results on one-dimensional maps with neutral fixed points.

Intermittent behaviour caused by the presence of neutral fixed points has also been considered for random dynamical systems. See e.g. [7, 6, 23, 8, 9] for results in this direction. The recent papers [2, 21] analysed another type of intermittent behaviour in random dynamical systems, so called *critical intermittency*. To illustrate the concept, consider the system generated by i.i.d. compositions of the logistic maps $T_2(x) = 2x(1-x)$ and $T_4(x) = 4x(1-x)$. The dynamics of these two maps individually is quite different: T_4 exhibits chaotic behaviour and admits an ergodic absolutely continuous invariant probability measure and T_2 has $\frac{1}{2}$ as a superattracting fixed point with $(0,1)$ as its basin of attraction. Since $T_4(\frac{1}{2}) = 1$ and $T_4(1) = T_2(1) = 0$, under random compositions of T_2 and T_4 the typical behaviour is the following: Orbits are quickly attracted to $\frac{1}{2}$ by applications of T_2 and are then repelled first close to 1 and then close to 0 by one application of T_4 followed by an application of either T_2 or T_4. Since 0 is a repelling fixed point, orbits then leave a neighbourhood of 0 after a number of applications of any combination of T_2 and T_4. See Figure 1. This pattern occurs infinitely often in the random orbits. The divergence from 0

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under T_2 and T_4 is exponential, whereas the convergence to $\frac{1}{2}$ under T_2 is superexponential and one can easily verify that $|T_2^k(x) - \frac{1}{2}| = \frac{1}{2}(2|\frac{1}{2} - x|)^{2^k}$. Consequently, for k sufficiently large, after $T_4 T_2^k$ is applied to a point in $T_4^{-1}[\frac{1}{2}, 1]$, it takes of the order 2^k time steps for any further realisation of this orbit to return to $T_4^{-1}[\frac{1}{2}, 1]$. In other words, there are two competing forces in play: superexponential attraction to $\frac{1}{2}$ and then to 0 and exponential repulsion away from 0.

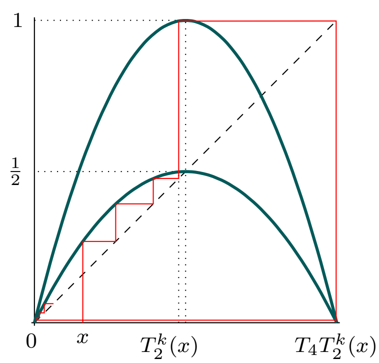


FIGURE 1. Critical intermittency in the random system of logistic maps T_2, T_4. CriticalInte

The authors of [2, 21] investigated the existence and finiteness of absolutely continuous invariant measures for certain random systems with critically intermittent behaviour, more in particular of random combinations of the two logistic maps T_2 and T_4 and of iterated function systems consisting of rational maps on the Riemann sphere. The long term behaviour of random iterations of logistic maps was also considered in [4, 5]. One particular result from [2] states that the random dynamical system generated by i.i.d. compositions of T_2 and T_4 chosen with probabilities p_2 and $p_4 = 1 - p_2$ admits an absolutely continuous invariant measure that is σ-finite on the interval $[0, 1]$ and that is infinite in case $p_2 > \frac{1}{2}$. An interesting question that was left open in [2] is whether for $p_2 \leq \frac{1}{2}$ this measure is infinite or finite.

In this article we answer this question. We consider a large family of random interval maps with critical intermittency that includes the random combination of T_2 and T_4. The systems we consider consist of i.i.d. compositions of a finite number of maps of two types: bad maps which share a superattracting fixed point and good maps that map the superattacting fixed point onto a common repelling fixed point. To be precise, the families of maps we consider are defined as follows.

Fix a point $c \in (0, 1)$. Throughout the text c will be fixed and it will represent the single critical point of our maps. A map $T_g : [0, 1] \to [0, 1]$ is in the class of *good maps*, denoted by $\mathfrak{G}$, if

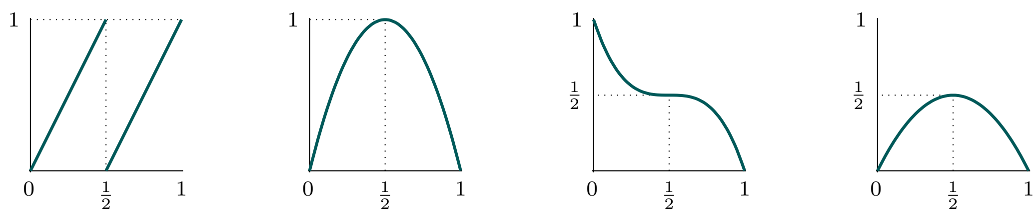
(G1) $T_g|_{[0,c)}$ and $T_g|_{(c,1]}$ are C^3 diffeomorphisms;
(G2) T_g has non-positive Schwarzian derivative on $[0, c)$ and $(c, 1]$;

(G3) $T_g(\{0,c,1\}) \subseteq \{0,1\}$ and $\lim_{x\uparrow c} T_g(x) \in \{0,1\}$ and $\lim_{x\downarrow c} T_g(x) \in \{0,1\}$;
(G4) to T_g we can associate three constants $r_g \geq 1$, $0 < K_g < 1$ and $M_g > r_g$ such that

eqn2.1 (1.1) $$K_g|x-c|^{r_g-1} \leq |DT_g(x)| \leq M_g|x-c|^{r_g-1};$$

(G5) we have $|DT_g(0)|, |DT_g(1)| > 1$.

These conditions imply in particular that at least one of the maps $T_g|_{[0,c]}$ or $T_g|_{[c,1]}$ is continuous, and that both branches of T_g are strictly monotone and their ranges contain $(0,1)$. Note also that the conditions $K_g < 1$ and $M_g > r_g$ are superfluous, since we can always choose a smaller constant K and larger constant M to satisfy (1.1), but we need these specific bounds in our estimates later. The critical point c is mapped to either 0 or 1 under each of the good maps and both 0 and 1 are (eventually) fixed points or periodic points (with period 2) by (G3) that are repelling by (G5). Examples include the doubling map and any surjective unimodal map, see Figures 2(a) and (b).



(a) $x \mapsto 2x \pmod 1$ (b) $x \mapsto 4x(1-x)$ (c) $x \mapsto \frac{1}{2} - 4(x-\frac{1}{2})^3$ (d) $x \mapsto 2x(1-x)$

T:examples FIGURE 2. Four maps with critical point $c = \frac{1}{2}$. (a) and (b) show two good maps, while in (c) and (d) we see the graphs of two bad maps.

A map $T_b : [0,1] \to [0,1]$ is in the class of *bad maps*, denoted by $\mathfrak{B}$, if

(B1) T_b is continuous and $T_b|_{[0,c)}$ and $T_b|_{(c,1]}$ are C^3 diffeomorphisms;
(B2) T_b has non-positive Schwarzian derivative on $[0,c)$ and $(c,1]$;
(B3) $T_b(\{0,1\}) \subseteq \{0,1\}$ and $T_b(c) = c$;
(B4) to T_b we can associate three constants $\ell_b > 1$, $0 < K_b < 1$ and $M_b > \ell_b$ such that

eqn2.2 (1.2) $$K_b|x-c|^{\ell_b-1} \leq |DT_b(x)| \leq M_b|x-c|^{\ell_b-1};$$

(B5) we have $|DT_b(0)|, |DT_b(1)| > 1$.

In particular (B1) implies that each map T_b is strictly monotone on the intervals $[0,c]$ and $[c,1]$. In contrast to (G4), note that in (B4) we have assumed that ℓ_b is not equal to one. This means that $DT_b(c) = 0$, so c is a superattracting fixed point for each bad map. For examples, see Figures 2(c) and (d).

The random systems we consider in this article are the following. Let $T_1, \ldots, T_N \in \mathfrak{G} \cup \mathfrak{B}$ be a finite collection of good and bad maps. Write $\Sigma_G = \{1 \leq j \leq N : T_j \in \mathfrak{G}\}$ and $\Sigma_B = \{1 \leq j \leq N : T_j \in \mathfrak{B}\}$ for the index sets of the good and bad maps respectively and assume that $\Sigma_G, \Sigma_B \neq \emptyset$. Write $\Sigma = \{1, \dots, N\} = \Sigma_G \cup \Sigma_B$. The *pseudo skew product transformation* or *random map* F is defined by

kewproduct
$$F : \Sigma^{\mathbb{N}} \times [0,1] \to \Sigma^{\mathbb{N}} \times [0,1],\ (\omega, x) \mapsto (\sigma\omega, T_{\omega_1}(x)), \tag{1.3}$$

where σ denotes the left shift on sequences in $\Sigma^{\mathbb{N}}$. Let $\mathbf{p} = (p_j)_{j\in\Sigma}$ be a probability vector representing the probabilities with which we choose the maps T_j, $j \in \Sigma$, and consider measures of the form $\mathbb{P} \times \mu_{\mathbf{p}}$, where $\mathbb{P}$ is the $\mathbf{p}$-Bernoulli measure on $\Sigma^{\mathbb{N}}$ and $\mu_{\mathbf{p}}$ is a Borel measure on $[0,1]$ absolutely continuous with respect to the one-dimensional Lebesgue measure λ and satisfying

eqn9
$$\sum_{j\in\Sigma} p_j \mu_{\mathbf{p}}(T_j^{-1}A) = \mu_{\mathbf{p}}(A), \qquad \text{for all Borel sets } A \subseteq [0,1]. \tag{1.4}$$

In this case $\mathbb{P} \times \mu_{\mathbf{p}}$ is an invariant measure for F and we say that $\mu_{\mathbf{p}}$ is a *stationary* measure for F. We also say that a stationary measure $\mu_{\mathbf{p}}$ is ergodic for F if $\mathbb{P} \times \mu_{\mathbf{p}}$ is ergodic for F. Our main results are the following.

MAIN **Theorem 1.1.** *Let $\{T_j : j \in \Sigma\}$ be as above and $\mathbf{p} = (p_j)_{j\in\Sigma}$ a positive probability vector.*

(1) There exists a unique (up to scalar multiplication) stationary σ-finite measure $\mu_{\mathbf{p}}$ for F that is absolutely continuous with respect to the one-dimensional Lebesgue measure λ. Moreover, this measure is ergodic.
(2) The density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is bounded away from zero, is locally Lipschitz on $(0,c)$ and $(c,1)$ and is not in L^q for any $q > 1$.

We call the measure $\mu_{\mathbf{p}}$ from Theorem 1.1 an *acs* measure.

MAIN2 **Theorem 1.2.** *Let $\{T_j : j \in \Sigma\}$ be as above and $\mathbf{p} = (p_j)_{j\in\Sigma}$ a positive probability vector. Let $\mu_{\mathbf{p}}$ be the unique acs measure from Theorem 1.1. Set $\theta = \sum_{b\in\Sigma_B} p_b \ell_b$. Then $\mu_{\mathbf{p}}$ is finite if and only if $\theta < 1$. In this case, there exists a constant $C > 0$ such that*

eq:2.10
$$\mu_{\mathbf{p}}(A) \leq C \cdot \sum_{k=0}^{\infty} \theta^k \lambda(A)^{\ell_{\max}^{-k} r_{\max}^{-1}} \tag{1.5}$$

for any Borel set $A \subseteq [0,1]$, where $r_{\max} = \max\{r_g : g \in \Sigma_G\}$ and $\ell_{\max} = \max\{\ell_b : b \in \Sigma_B\}$.

As we shall see in (4.12) the bound in (1.5) can be improved by not bounding mixtures $\ell_{\mathbf{b}} r_g = \prod_{i=1}^k \ell_{b_i} r_g$ by their maximal value $\ell_{\max}^k r_{\max}$.

It will become clear that the density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ in Theorem 1.1 blows up to infinity at the points zero and one and also (at least on one side) at c. Theorem 1.2 says that $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is integrable if and only if θ is small enough, namely $\theta < 1$. This intuitively makes sense since for a smaller value of θ the attraction of orbits to c is weaker on average and consequently orbits typically spend less time near zero and one once a good map is applied.

The acs measure $\mu = \mu_{\mathbf{p}}$ from Theorem 1.1 depends on the chosen probability vector $\mathbf{p} \in \mathbb{R}^N$. We have the following stability result.

cor **Corollary 1.1.** *Let $\{T_j : j \in \Sigma\}$ be as above. For each $n \geq 0$, let $\mathbf{p}_n = (p_{n,j})_{j\in\Sigma}$ be a positive probability vector such that $\sum_{j\in\Sigma_B} p_{n,j}\ell_j < 1$ and assume that $\lim_{n\to\infty} \mathbf{p}_n = \mathbf{p}_0$ in $\mathbb{R}^N$. Let μ_n be the unique acs probability measure for F corresponding to $\mathbf{p}_n$. Then the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ_0.*

In (B4) we have assumed that for any bad map T_b the corresponding value ℓ_b is not equal to one. Note that a bad map T_b for which we allow $\ell_b = 1$ satisfies $|DT_b(c)| > 0$, so in this case c is an attracting fixed point for T_b but not superattracting. It should not come as a surprise that results similar to Theorem 1.1 and Theorem 1.2 also hold in case some or all of the bad maps T_b have $\ell_b = 1$. The proofs presented for these theorems, however, do not immediately carry over. In the last section we explain how the results need to be modified in case some or all maps T_b satisfy $\ell_b = 1$ and what the necessary changes in the proofs are.

The proofs use a mixture of techniques. For the existence result from Theorem 1.1 we use an inducing scheme together with the Folklore Theorem. This is inspired by [2], but the choice of the inducing domain needed some care. With Kac's Lemma we then obtain that the acs measure is infinite in case $\theta \geq 1$. To prove that the this measure is finite for $\theta < 1$ we use an approach similar to the one employed in [28] for deterministic systems. To deal with the randomness in the system, however, we needed to improve the control on the pushforwards of the Lebesgue measure under all possible random combinations of the good and bad maps. The main difficulty here is that it may take an arbitrarily long time before the superattracting fixed point is mapped onto the repelling orbit by one of the good maps, which decreases the regularity of the density of the acs measure.

The paper is organised as follows. In Section 2 we list some preliminaries and first consequences of the conditions (G1)–(G5) and (B1)–(B5). Section 3 is devoted to the proof of Theorem 1.1 and in Section 4 we prove Theorem 1.2. In Section 5 we prove Corollary 1.1 and explain what the analogues of Theorem 1.1 and 1.2 are in case $\ell_b = 1$ for one or more $b \in \Sigma$ and how the proofs of Theorem 1.1 and 1.2 need to be modified to get these results. We end with some final remarks.

2. Preliminaries

sec:mr

We start by introducing some notation and collecting some general preliminaries.

2.1. **Words, sequences and invariant measures.**

For any finite subset $\Sigma \subseteq \mathbb{N}$ and any $n \geq 1$ we use $\mathbf{u} \in \Sigma^n$ to denote a *word* $\mathbf{u} = u_1 \cdots u_n$. Σ^0 contains only the empty word, which we denote by ϵ. On the space of infinite sequences $\Omega = \Sigma^{\mathbb{N}}$ we use

$$[\mathbf{u}] = [u_1 \cdots u_n] = \{\omega \in \Omega : \omega_1 = u_1, \dots, \omega_n = u_n\}$$

to denote the *cylinder set* corresponding to $\mathbf{u}$. The notation $|\mathbf{u}|$ indicates the length of $\mathbf{u}$, so $|\mathbf{u}| = n$ for $\mathbf{u} \in \Sigma^n$. For two words $\mathbf{u} \in \Sigma^n$ and $\mathbf{v} \in \Sigma^m$ the concatenation of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{uv} \in \Sigma^{n+m}$. For a probability vector $p = (p_j)_{j\in\Sigma}$ and $\mathbf{u} \in \Sigma^n$ we write $p_{\mathbf{u}} = \prod_{i=1}^{n} p_{u_i}$ with $p_{\mathbf{u}} = 0$ if $n = 0$. On Ω we use σ to denote the *left shift*, i.e., $(\sigma\omega)_n = \omega_{n+1}$.

Given a finite family of Borel measurable maps $\{T_j : X \to X\}_{j\in\Sigma}$ on a metric space X, the random map F is given by

$$F : \Omega \times X \to \Omega \times X,\ (\omega, x) \mapsto \big(\sigma\omega, T_{\omega_1}(x)\big).$$

We use the following notation for the iterates of the maps T_j. For each $\omega \in \Omega$ and each $n \in \mathbb{N}_0$ define

$$T_{\omega_1\cdots\omega_n}(x) = T^n_\omega(x) = \begin{cases} x & \text{if } \ n = 0,\\ T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}(x) & \text{for } n \geq 1.\end{cases} \tag{2.1}$$

With this notation, we can write the iterates of the random system F as

$$F^n(\omega, x) = (\sigma^n\omega, T^n_\omega(x)). \tag{2.2}$$

The following lemma on invariant measures for F holds.

uctmeasure **Lemma 2.1** ([27], see also Lemma 3.2 of [17]). *Let m denote the Lebesgue measure on X. If all maps T_j are non-singular with respect to m and $\mathbb{P}$ the $\mathbf{p}$-Bernoulli measure on Ω for some positive probability vector $\mathbf{p}$, then the $\mathbb{P} \times m$-absolutely continuous F-invariant measures are precisely the measures of the form $\mathbb{P} \times \mu$ where μ is absolutely continuous w.r.t. m and satisfies*

eqn9
$$\sum_{j\in\Sigma} p_j \mu(T_j^{-1}A) = \mu(A) \qquad \textit{for all Borel sets } A. \tag{2.3}$$

Now let $(X, \mathcal{F}, m)$ be a measure space and $T : X \to X$ a measurable and non-singular transformation. For a set $Y \in \mathcal{F}$ with $0 < m(Y) < \infty$ and $m\big(X \setminus \bigcup_{n\geq 0} T^{-n}Y\big) = 0$ the *first return time map* $\varphi_Y : Y \to \mathbb{N} \cup \{\infty\}$ given by

q:frtm
$$\varphi_Y(y) = \inf\{n \geq 1 : T^n(y) \in Y\} \tag{2.4}$$

is finite for m-a.e. $y \in Y$ and moreover m-a.e. $y \in Y$ returns to Y i.o. If we remove from Y the m-null set of points that return to Y only finitely many times and for convenience call this set Y again, then we can define the *induced transformation* $T_Y : Y \to Y$ by

$$T_Y(y) = T^{\varphi_Y(y)}(y).$$

The following result can be found in e.g. [1, Proposition 1.5.7]. Note that this statement asks for T to be conservative. This is not used in the proof however and the condition $m\big(X \setminus \bigcup_{n\geq 0} T^{-n}Y\big) = 0$ is enough to guarantee that the induced transformation is well defined.

cedmeasure **Lemma 2.2.** *Let T be a measurable and non-singular transformation on a measure space $(X, \mathcal{F}, m)$ and let $Y \in \mathcal{F}$ be such that $0 < m(Y) < \infty$ and $m\big(X \setminus \bigcup_{n\geq 0} T^{-n}Y\big) = 0$. If*

$\nu \ll m|_Y$ is a finite invariant measure for the induced transformation T_Y, then the measure μ on $(X, \mathcal{F}, m)$ defined by

$$\mu(B) = \sum_{k \geq 0} \nu\Big(Y \cap T^{-k} B \setminus \bigcup_{j=1}^{k} T^{-j} Y \Big)$$

for $B \in \mathcal{F}$ is T-invariant, absolutely continuous with respect to m and $\mu|_Y = \nu$.

We will also use the following result on the first return time.

l:kac **Lemma 2.3** (Kac's Formula, see e.g. 1.5.5. in [1])**.** *Let T be a conservative, ergodic, measure preserving transformation on a measure space $(X, \mathcal{F}, m)$. Let $Y \in \mathcal{F}$ be such that $0 < m(Y) < \infty$ and let φ_Y be the first return map to Y. Then $\int_Y \varphi_Y \, dm = m(X)$.*

One can also obtain invariant measures via a functional analytic approach. Here we give a specific result for interval maps. Let I be an interval and, as before, let λ denote the one-dimensional Lebesgue measure on I. If $T : I \to I$ is piecewise strictly monotone and C^1, then the *Perron-Frobenius operator* $\mathcal{P}_T : L^1(I, \lambda) \to L^1(I, \lambda)$ is defined by

q:pfd
$$\mathcal{P}_T h(x) = \sum_{y \in T^{-1}\{x\}} \frac{h(y)}{|DT(y)|}. \tag{2.5}$$

A non-negative function $\varphi \in L^1(I, \lambda)$ is a fixed point of $\mathcal{P}$ if and only if it provides an invariant measure μ for T that is absolutely continuous with respect to λ by setting $\mu(A) = \int_A \varphi \, d\lambda$ for each Borel set A.

For a random system F using a finite family of transformations $\{T_j : I \to I\}_{j \in \Sigma}$, such that each map T_j is piecewise strictly monotone and C^1, and a positive probability vector $\mathbf{p} = (p_j)_{j \in \Sigma}$, the Perron-Frobenius operator $\mathcal{P}_F$ on the space of non-negative measurable functions h on $[0, 1]$ is given by

eqn3.22
$$\mathcal{P}_F h(x) = \sum_{j \in \Sigma} p_j \mathcal{P}_{T_j} h(x), \tag{2.6}$$

where each $\mathcal{P}_{T_j}$ is as given in (2.5). Let $\mathbb{P}$ denote the $\mathbf{p}$-Bernoulli measure on Ω. Then a non-negative function $\varphi \in L^1(I, \lambda)$ is a fixed point of $\mathcal{P}_F$ if and only if the measure $\mathbb{P} \times \mu$, where μ is the absolutely continuous measure with $\frac{d\mu}{d\lambda} = \varphi$, is F-invariant.

In Subsection 3.3 it will be shown that the density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ from Theorem 1.1, which is a fixed point of the Perron-Frobenius operator for the random system F given by (1.3), is bounded away from zero. From this it is easy to see that (2.6) implies that $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ blows up to infinity at the points zero and one and also at least on one side of c.

2.2. **Estimates on good and bad maps.** Now let $T : I \to I$ be a C^3 map of an interval I into itself. The *Schwarzian derivative* of T at $x \in I$ with $DT(x) \neq 0$ is defined by

$$\mathbf{S}T(x) = \frac{D^3 T(x)}{DT(x)} - \frac{3}{2}\Big(\frac{D^2 T(x)}{DT(x)} \Big)^2. \tag{2.7}$$

We say that T *has non-positive Schwarzian derivative* on I if $DT(x) \neq 0$ and $\mathbf{S}T(x) \leq 0$ for all $x \in I$. A direct computation shows that the Schwarzian derivative of the composition of two transformations $T_1, T_2 : I \to I$ satisfies

eq4
$$\mathbf{S}(T_2 \circ T_1)(x) = \mathbf{S}T_2\big(T_1(x)\big) \cdot |DT_1(x)|^2 + \mathbf{S}T_1(x). \tag{2.8}$$

Hence, $\mathbf{S}(T_2 \circ T_1) \leq 0$ provided $\mathbf{S}T_1 \leq 0$ and $\mathbf{S}T_2 \leq 0$.

From (2.8) it follows that for a finite collection $\{T_j : I \to I\}_{j \in \Sigma}$ of C^3 interval maps with non-positive Schwarzian derivative, we can write the Schwarzian derivative of T_ω^n, $n \in \mathbb{N}$ and $\omega \in \Omega$, as

eqn5
$$\mathbf{S}T_\omega^n(x) = \sum_{i=0}^{n-1} \mathbf{S}T_{\omega_{i+1}}\big(T_\omega^i(x)\big) \cdot \Big| \prod_{j=1}^{i} DT_{\omega_j}(T_\omega^{j-1}(x)) \Big|^2. \tag{2.9}$$

By (G2) and (B2) this implies that for a collection of good and bad maps $\{T_j\}_{j \in \Sigma}$, T_ω^n has non-positive Schwarzian derivative on $[0,1]$ outside of the critical points of T_ω^n for all $\omega \in \Omega$ and $n \in \mathbb{N}$.

We will use the following two well-known properties of maps with non-positive Schwarzian derivative (see e.g. [15, Section 4.1]).

Minimum Principle: Let $I = [a,b]$ be a closed interval and suppose that $T : I \to I$ has non-positive Schwarzian derivative. Then

$$|DT(x)| \geq \min\{DT(a), DT(b)\}, \quad \forall x \in [a,b]. \tag{2.10}$$

As a consequence of the Minimum Principle for any $T \in \mathfrak{G} \cup \mathfrak{B}$ the derivative $|DT|$ has locally no strict minima in the intervals $(0,c)$ and $(c,1)$. In particular, there cannot be any attracting fixed points for T in $(0,c)$ and $(c,1)$. Therefore, if $T \in \mathfrak{B}$, then $T^n(x) \to c$ as $n \to \infty$ for all $x \in (0,1)$.

Koebe Principle: For each $\rho > 0$ there exist $K^{(\rho)} > 1$ and $M^{(\rho)} > 0$ with the following property. Let $J \subseteq I$ be two intervals and suppose that $T : I \to I$ has non-positive Schwarzian derivative. If both components of $T(I) \backslash T(J)$ have length at least $\rho \cdot \lambda(T(J))$, then

eq2.4
$$\frac{1}{K^{(\rho)}} \leq \frac{DT(x)}{DT(y)} \leq K^{(\rho)}, \qquad \forall x, y \in J \tag{2.11}$$

and

eq2.5
$$\Big| \frac{DT(x)}{DT(y)} - 1 \Big| \leq M^{(\rho)} \cdot \frac{|T(x) - T(y)|}{\lambda(T(J))}, \qquad \forall x, y \in J. \tag{2.12}$$

Note that the constants $K^{(\rho)}, M^{(\rho)}$ only depend on ρ and not on the map T.

From (2.11) one can obtain a bound on the size of the images of intervals: Let $J' \subseteq J$ be another interval. By the Mean Value Theorem there exists an $x \in J'$ with $|DT(x)| =$

$\frac{\lambda(T(J'))}{\lambda(J')}$ and a $y \in J$ with $|DT(y)| = \frac{\lambda(T(J))}{\lambda(J)}$. Hence,

:intervals

$$\frac{1}{K^{(\rho)}} \frac{\lambda(J')}{\lambda(J)} \leq \frac{DT(x)}{DT(y)} \frac{\lambda(J')}{\lambda(J)} = \frac{\lambda(T(J'))}{\lambda(T(J))} \leq K^{(\rho)} \frac{\lambda(J')}{\lambda(J)}. \tag{2.13}$$

Recall the constants ℓ_b, K_b and M_b from condition (B4) and set $\ell_{\min} = \min\{\ell_b : b \in \Sigma_B\}$ and $\ell_{\max} = \max\{\ell_b : b \in \Sigma_B\}$. (B4) gives us control over the distance between $T_\omega^n(x)$ and c.

lemma3.6

Lemma 2.4. *For all $n \in \mathbb{N}$, $\omega \in \Sigma_B^{\mathbb{N}}$ and $x \in [0,1]$,*

$$\Big(\tilde{K}|x-c|\Big)^{\ell_{\omega_1}\cdots\ell_{\omega_n}} \leq |T_\omega^n(x) - c| \leq \Big(\tilde{M}|x-c|\Big)^{\ell_{\omega_1}\cdots\ell_{\omega_n}}, \tag{2.14}$$

with constants $\tilde{K} = \big(\frac{\min\{K_b : b \in \Sigma_B\}}{\ell_{\max}}\big)^{\frac{1}{\ell_{\min}-1}} \in (0,1)$ and $\tilde{M} = \big(\frac{\max\{M_b : b\in\Sigma_B\}}{\ell_{\min}}\big)^{\frac{1}{\ell_{\min}-1}} > 1$.

Proof. It follows from (B4) that for any $j \in \Sigma_B$ and $x \in [0,1]$,

$$|T_j(x) - c| = |T_j(x) - T_j(c)| = \Big| \int_c^x DT_j(y)dy \Big| \geq \frac{\min\{K_b : b \in \Sigma_B\}}{\ell_{\max}} |x-c|^{\ell_j}.$$

By induction we get that for each $n \in \mathbb{N}$ and $\omega \in \Sigma_B^{\mathbb{N}}$,

eq3.29

$$|T_\omega^n(x) - c| \geq \left(\frac{\min\{K_b : b \in \Sigma_B\}}{\ell_{\max}} \right)^{1+\sum_{i=0}^{n-2} \ell_{\omega_n}\cdots\ell_{\omega_{n-i}}} \cdot |x-c|^{\ell_{\omega_1}\cdots\ell_{\omega_n}}. \tag{2.15}$$

From (B4) we see that $\frac{\min\{K_b : b\in\Sigma_B\}}{\ell_{\max}} < 1$. The lower bound now follows by observing that

$$\Big(1 + \sum_{i=0}^{n-2} \ell_{\omega_n} \cdots \ell_{\omega_{n-i}}\Big) / (\ell_{\omega_1} \cdots \ell_{\omega_n}) \leq \sum_{i=1}^{n} \frac{1}{\ell_{\min}^i} < \frac{1}{\ell_{\min} - 1}.$$

The result for the upper bound follows similarly, by noticing that in this case from (B4) it follows that $\frac{\max\{M_b : b\in\Sigma_B\}}{\ell_{\min}} > 1$. □

It follows that under iterations of bad maps the distance $|T_\omega^n(x) - c|$ is eventually decreasing superexponentially fast in n with an increasing rate that grows itself exponentially fast in n. Furthermore, taking x in a sufficiently small neighbourhood of c the distance $|T_\omega^n(x) - c|$ is strictly decreasing as n increases. Indeed, note that there exists $\delta > 0$ such that $|DT_b(x)| < 1$ for all $x \in [c-\delta, c+\delta]$ and $b \in \Sigma_B$. This implies

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$$|T_b(x) - c| < |x - c| \tag{2.16}$$

for all $x \in [c-\delta, c+\delta]$ and $b \in \Sigma_B$.

The upper bound on $|T_\omega^n(x) - c|$ that we obtained in Lemma 2.4 will be used in Section 4 to prove that $\mu_{\mathbf{p}}$ in Theorem 1.2 is infinite if $\theta \geq 1$. The lower bound from Lemma 2.4 will be used for the proof that $\mu_{\mathbf{p}}$ is finite if $\theta < 1$.

sec:PT2.1

3. Existence of a σ-finite acs measure

From now on we fix an integer $N \geq 2$ and consider a finite collection $T_1, \dots, T_N \in \mathfrak{G} \cup \mathfrak{B}$ of good and bad maps in the classes $\mathfrak{G}$ and $\mathfrak{B}$. As in the Introduction write $\Sigma_G = \{1 \leq j \leq N : T_j \in \mathfrak{G}\}$ and $\Sigma_B = \{1 \leq j \leq N : T_j \in \mathfrak{B}\}$ for the corresponding index sets and assume that $\Sigma_G, \Sigma_B \neq \emptyset$. Write $\Sigma = \{1, 2, \dots, N\}$ and set $\Omega = \Sigma^{\mathbb{N}}$ for the set of infinite sequences of elements in Σ. In this section we prove Theorem 1.1, i.e., the existence of an ergodic acs measure with several properties for the random system F by using an inducing scheme. We fix the index $g \in \Sigma_G$ of one good map T_g and start by constructing the inducing domain.

3.1. **The induced system and return time partition.** The first lemma is needed to specify the set on which we induce. For each $k \in \mathbb{N}$ let x_k and x'_k in $(0, c)$ denote the critical points of T_g^k closest to 0 and c, respectively. Furthermore, let y_k and y'_k in $(c, 1)$ denote the critical points of T_g^k closest to 1 and c, respectively.

riticalpts **Lemma 3.1.** *We have $x_k \downarrow 0$, $x'_k \uparrow c$, $y'_k \downarrow c$, $y_k \uparrow 1$ as $k \to \infty$.*

Proof. Let a and b denote the critical points of T_g^2 in $(0, c)$ and $(c, 1)$, respectively. Then at least one of the branches $T_g^2|_{(0,a)}$ and $T_g^2|_{(b,1)}$ is increasing. Suppose that $T_g^2|_{(0,a)}$ is increasing. It then follows from the Minimum Principle that $T_g^2(x) \geq \min\{\frac{x}{a}, DT_g^2(0) \cdot x\}$ for each $x \in [0, a]$. To see this, suppose there is an $x \in (0, a)$ with $T_g^2(x) < \min\{\frac{x}{a}, DT_g^2(0) \cdot x\}$. Then there must be a $y \in (0, x)$ with $DT_g^2(y) < \min\{DT_g^2(0), \frac{1}{a}\}$ and a $z \in [x, a)$ with $DT_g^2(z) > \frac{1}{a}$. On the other hand, by the Minimum Principle, $DT_g^2(y) \geq \min\{DT_g^2(0), DT_g^2(z)\}$, a contradiction. Combining this with $DT_g^2(0) > 1$ and defining $L : (0, 1) \to (0, a)$ by $L = (T_g^2|_{(0,a)})^{-1}$, we see that $L^k(a) \downarrow 0$ as $k \to \infty$. Furthermore, define $R : (0, 1) \to (b, 1)$ by $R = (T_g^2|_{(b,1)})^{-1}$. If $T_g^2|_{(b,1)}$ is increasing, we see that similarly $R^k(b) \uparrow 1$ as $k \to \infty$. On the other hand, if $T_g^2|_{(b,1)}$ is decreasing, we have $RL^k(a) \uparrow 1$ as $k \to \infty$. Finally, if $T_g^2|_{(0,a)}$ is decreasing, then $T_g^2|_{(b,1)}$ must be increasing, which yields $LR^k(b) \downarrow 0$ as $k \to \infty$. We conclude that $x_k \downarrow 0$ and $y_k \uparrow 1$ as $k \to \infty$. It follows from (G3) that c is a limit point of both of the sets $\bigcup_{k\in\mathbb{N}} (T_g|_{(0,c)})^{-1}(\{x_k, y_k\})$ and $\bigcup_{k\in\mathbb{N}} (T_g|_{(c,1)})^{-1}(\{x_k, y_k\})$. So $x'_k \uparrow c$, $y'_k \downarrow c$ as $k \to \infty$. □

By the previous lemma, (G1) and (G3), for $k \in \mathbb{N}$ large enough it holds that

eq:3.1
$$T_g(x'_k) \leq x'_k \text{ or } T_g(x'_k) \geq y'_k, \text{ and} \tag{3.1}$$

eq:3.2
$$T_g(y'_k) \leq x'_k \text{ or } T_g(y'_k) \geq y'_k, \tag{3.2}$$

and, using also (G5), (B1), (B3) and (B5), for every $j \in \Sigma$,

eq:3.3
$$T_j\big([0, x_k] \cup [y_k, 1]\big) \subseteq [0, x'_k) \cup (y'_k, 1] \text{ and } |DT_j(x)| > d > 1 \text{ for all } x \in [0, x_k) \cup (y_k, 1] \tag{3.3}$$

and some constant d. Fix a $\kappa \in \mathbb{N}$ for which (3.1), (3.2) and (3.3) hold. We introduce some notation. Let $t \in \Sigma$ be such that $t \neq g$, and define

$$C = [\underbrace{g \cdots g}_{\kappa \text{ times}} t] = [g^\kappa t], \tag{3.4}$$

$$J_0 = (x_\kappa, x'_\kappa), \quad J_1 = (y'_\kappa, y_\kappa), \quad J = J_0 \cup J_1, \tag{3.5}$$

$$Y = C \times J. \tag{3.6}$$

The next lemma states that $\mathbb{P} \times \lambda$-almost all (ω, x) eventually enter Y under iterations of F, so that $\mathbb{P} \times \lambda$-almost all $(\omega, x) \in Y$ will return to Y infinitely many times.

lemma3.2 **Lemma 3.2.**

$$\mathbb{P} \times \lambda\Big(\Omega \times [0,1] \setminus \bigcup_{n=1}^{\infty} F^{-n} Y\Big) = 0. \tag{3.7}$$

Proof. For $\mathbb{P}$-almost all $\omega \in \Omega$ we have $\sigma^n \omega \in [g]$ for infinitely many $n \in \mathbb{N}$. For any such n either $T_\omega^n(x) \in J$ or $T_\omega^n(x) \notin J$. If $T_\omega^n(x) \in (0, x_\kappa] \cup [y_\kappa, 1)$, then it follows from (3.3) that there is an $m \geq 1$ such that $T_\omega^{n+m}(x) \in J$. If $T_\omega^n(x) \in [x'_\kappa, c) \cup (c, y'_\kappa]$ it follows from (3.1) and (3.2) that $T_\omega^{n+1}(x) = T_g \circ T_\omega^n(x) \in (0, x'_\kappa] \cup [y'_\kappa, 1)$, which means that we are in the first case if $T_\omega^{n+1}(x) \notin J$. Hence, for $\mathbb{P} \times \lambda$-almost all $(\omega, x) \in \Omega \times [0,1]$ we have $T_\omega^n(x) \in J$ for infinitely many $n \in \mathbb{N}$. Consider such an (ω, x), and let $(n_j)_{j \in \mathbb{N}}$ be an increasing sequence in $\mathbb{N}$ that satisfies $T_\omega^{n_j} x \in J$ for each $j \in \mathbb{N}$. Recall that σ denotes the left shift on sequences and define $\mathcal{E} = \Sigma \backslash (\bigcup_{j \in \mathbb{N}} \sigma^{-n_j} C)$, i.e., $\mathcal{E}$ contains precisely those sequences ω' that satisfy $\sigma^{n_j}(\omega') \notin C$ for all n_j. According to the Lebesgue Differentiation Theorem (see e.g. [33]) we may assume that ω is a Lebesgue point of $1_\mathcal{E}$, which yields

$$\begin{aligned} 1 \geq \frac{\mathbb{P}\big((\mathcal{E} \cup \sigma^{-n_j} C) \cap [\omega_1 \cdots \omega_{n_j}]\big)}{\mathbb{P}\big([\omega_1 \cdots \omega_{n_j}]\big)} &= \frac{\mathbb{P}\big(\mathcal{E} \cap [\omega_1 \cdots \omega_{n_j}]\big)}{\mathbb{P}\big([\omega_1 \cdots \omega_{n_j}]\big)} + \frac{\mathbb{P}\big(\sigma^{-n_j} C \cap [\omega_1 \cdots \omega_{n_j}]\big)}{\mathbb{P}\big([\omega_1 \cdots \omega_{n_j}]\big)} \\ &\to 1_\mathcal{E}(\omega) + \mathbb{P}(C), \qquad \text{as } j \to \infty. \end{aligned}$$

Since $\mathbb{P}(C) > 0$, we conclude that $\omega \notin \mathcal{E}$. Hence, there is an n_j so that $F^{n_j}(\omega, x) \in C \times J = Y$. □

By Lemma 3.2 the first return time map φ_Y, see (2.4), and the induced transformation F_Y are well defined on the full measure subset of points in Y that return to Y infinitely often under iterations of F, which we call Y again. The set of points in Y that return to Y after n iterations of F can be described as

q:leveln

$$Y \cap F^{-n}(Y) = \bigcup_{\omega \in C \cap \sigma^{-n} C} [\omega_1 \cdots \omega_n] \times (T_\omega^n|_J)^{-1}(J) \mod \mathbb{P} \times \lambda, \tag{3.8}$$

which is empty for $n \leq \kappa$. Note that in (3.8) in fact $[\omega_1 \cdots \omega_n] = [g^\kappa t \omega_{k+2} \cdots \omega_n g^\kappa t]$ and that by construction each map $T_\omega^n|_J$ in (3.8) consists of branches that all have range $(0, c)$ or $(c, 1)$ or $(0, 1)$, since any branch of $T_\omega^\kappa|_J$ maps onto $(0, 1)$. Therefore, $Y \cap F^{-n}(Y)$ can be written as a finite union of products $A = [\mathbf{u} g^\kappa t] \times I$ of cylinders $[\mathbf{u} g^\kappa t] \subseteq C$ with $|\mathbf{u}| = n$ and open intervals $I \subseteq J$, each of which is mapped under F^n onto $C \times J_0$ or $C \times J_1$. Call the collection of these sets P_n and let $\alpha = \bigcup_{n>\kappa} P_n$. Let $\mathbb{P}_C$ and λ_J denote the normalized restrictions of $\mathbb{P}$ to C and λ to J respectively.

:disjointp **Lemma 3.3.**

(1) *The collection α forms a countable* return time partition *of Y, i.e., $\mathbb{P}_C \times \lambda_J(\bigcup_{A\in\alpha} A) = 1$, any two different sets $A, A' \in \alpha$ are disjoint and on any $A \in \alpha$ the first return time map φ_Y is constant.*

(2) *Let π denote the canonical projection onto the second coordinate. Any $x \in J$ is contained in a set $\pi(A)$ for some set $A \in \alpha$.*

Proof. The fact that $\mathbb{P}_C \times \lambda_J(\bigcup_{A\in\alpha} A) = 1$ follows from Lemma 3.2 and it is clear from the construction that the first return time map φ_Y is constant on any element $A \in \alpha$. To show that any two elements are disjoint, note that for $A, A' \in P_n$ this is clear. Suppose there are $1 \le m < n$, $A = [\mathbf{u}g^\kappa t] \times I \in P_n$ and $A' = [\mathbf{v}g^\kappa t] \times I' \in P_m$ such that $A \cap A' \neq \emptyset$. Since $t \neq g$ we get $n \ge m + \kappa + 1$ and $[\mathbf{u}g^\kappa t] = [g^\kappa t v_{\kappa+2} \cdots v_m g^\kappa t u_{m+\kappa+2} \cdots u_n g^\kappa t]$. Moreover, $I \cap \partial I' \neq \emptyset$ or $I = I'$. In both cases, note that $F^{m+\kappa+1}([\mathbf{v}g^\kappa t] \times \partial I') \subseteq \Omega \times \{0,1\}$, so by (G3) and (B3) also $F^n([\mathbf{v}g^\kappa t] \times \partial I') \subseteq \Omega \times \{0,1\}$, contradicting that $F^n(A) \subseteq Y$. This proves (1).

For (2), let $A = [\mathbf{u}g^\kappa t] \times I \in \alpha$ and $x \in \partial I$. Then $T_\mathbf{u}(x) \in \partial J_i$ for some $i \in \{0,1\}$. From the first part of the proof of Lemma 3.2 it then follows that there is an $n > |\mathbf{u}|$ and an $\omega \in C$ such that $T^n_\omega(x) \in J$. This means that there exists a set $[\mathbf{v}g^\kappa t] \times I' \in \alpha$ so that $x \in I'$. □

The second part of Lemma 3.3 shows that even though the partition elements of α are disjoint, their projections on the second coordinate are not. The same is true for the first coordinate as the same string $\mathbf{u}$ can lead points in J to J_0 and J_1.

3.2. **Properties of the induced transformation.** It follows from (3.8) and Lemma 3.3 that for each $A \in \alpha$ we have either $F_Y(A) = C \times J_0$ or $F_Y(A) = C \times J_1$. For any $[\mathbf{u}g^\kappa t] \times I \in \alpha$, the transformation $T_\mathbf{u}|_I$ is invertible from I to one of the sets J_0 or J_1. Define the operator $\mathcal{P}_{\mathbf{u},I} : L^1(J, \lambda_J) \to L^1(J, \lambda_J)$ by

$$\mathcal{P}_{\mathbf{u},I} h(x) = \begin{cases} \dfrac{h(T_\mathbf{u}|_I^{-1}(x))}{\left|DT_\mathbf{u}|_I(T_\mathbf{u}|_I^{-1}(x))\right|}, & \text{if } T_\mathbf{u}|_I^{-1}\{x\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

The *random Perron-Frobenius-type operator* $\mathcal{P}_Y : L^1(J, \lambda_J) \to L^1(J, \lambda_J)$ on Y is given by

eqn3.12
$$\mathcal{P}_Y = \sum_{[\mathbf{u}g^\kappa t]\times I \in \alpha} \mathbb{P}_C([\mathbf{u}]) \mathcal{P}_{\mathbf{u},I}. \tag{3.10}$$

Note that $\mathcal{P}_Y$ is not exactly of the same form as the usual Perron-Frobenius operator in (2.6). Nonetheless, we have the following result.

l:pflike **Lemma 3.4.** *If $\varphi \in L^1(J, \lambda_J)$ is a fixed point of $\mathcal{P}_Y$, then the measure $\mathbb{P}_C \times \nu$ with $\nu = \varphi d\lambda_J$ is invariant for F_Y.*

Proof. For each cylinder $K \subseteq C$ and each Borel set $E \subseteq J$ we have

$$\begin{aligned}\mathbb{P}_C \times \nu\big(F_Y^{-1}(K \times E)\big) &= \sum_{[\mathbf{u}g^\kappa t]\times I \in \alpha} \mathbb{P}_C([\mathbf{u}g^\kappa t] \cap \sigma^{-|\mathbf{u}|}K)\nu(I \cap T_{\mathbf{u}}^{-1}E) \\ &= \mathbb{P}_C(K) \sum_{[\mathbf{u}g^\kappa t]\times I \in \alpha} \mathbb{P}_C([\mathbf{u}]) \int_E \mathcal{P}_{\mathbf{u},I}\varphi d\lambda_J \\ &= \mathbb{P}_C(K) \int_E \mathcal{P}_Y \varphi d\lambda_J \\ &= \mathbb{P}_C \times \nu(K \times E). \qquad \square\end{aligned}$$

In Lemma 3.5 below we show that a fixed point of $\mathcal{P}_Y$ exists. For $m \in \mathbb{N}$, set $\alpha_m = \bigvee_{j=0}^{m-1} F_Y^{-j}\alpha$. Atoms of this partition are the *m-cylinders of F_Y*. Introducing for each $Z = \bigcap_{j=0}^{m-1} F_Y^{-j}([\mathbf{u}_j g^\kappa t] \times I_j)$ in α_m the notation

q:czjz (3.11)
$$C_Z = \bigcap_{j=0}^{m-1} \sigma^{-\sum_{i=0}^{j-1}|\mathbf{u}_i|}[\mathbf{u}_j g^\kappa t] \quad \text{and} \quad J_Z = \bigcap_{j=0}^{m-1} T^{-1}_{\mathbf{u}_0\mathbf{u}_1\cdots\mathbf{u}_{j-1}}(I_j), \tag{3.11}$$

we obtain $Z = C_Z \times J_Z$. Writing $\sigma_Z = \sigma^{\sum_{i=0}^{m-1}|\mathbf{u}_i|}|_{C_Z}$ and $T_Z = T_{\mathbf{u}_0\mathbf{u}_1\cdots\mathbf{u}_{m-1}}|_{J_Z}$ we have $F_Y^m|_Z = \sigma_Z \times T_Z$. Each T_Z has non-positive Schwarzian derivative, so we can apply the Koebe Principle. The image $T_Z(J_Z)$ either equals J_0 or J_1. Choose a $\bar\rho > 0$ such that $I_0 := [x_\kappa - \bar\rho, x'_\kappa + \bar\rho] \subseteq (0, c)$ and $I_1 := [y'_\kappa - \bar\rho, y_\kappa + \bar\rho] \subseteq (c, 1)$. There is a canonical way to extend the domain of each T_Z to an interval I containing J_Z, such that $T_Z(I)$ equals either I_0 or I_1 and $\mathbf{S}(T_Z) \le 0$ on I. Then by the Koebe Principle there exist constants $K^{(\bar\rho)} > 1$ and $M^{(\bar\rho)} > 0$ such that for all $m \in \mathbb{N}$, $Z \in \alpha_m$ and $x, y \in J_Z$,

eqn3.16a (3.12)
$$\frac{1}{K^{(\bar\rho)}} \le \frac{DT_Z(x)}{DT_Z(y)} \le K^{(\bar\rho)}, \tag{3.12}$$

eqn3.16 (3.13)
$$\Big|\frac{DT_Z(x)}{DT_Z(y)} - 1\Big| \le \frac{M^{(\bar\rho)}}{\min\{\lambda(I_0), \lambda(I_1)\}} \cdot |T_Z(x) - T_Z(y)|. \tag{3.13}$$

Note that for the random Perron-Frobenius-type operator from (3.10) we have for each $m \ge 1$ that

$$\mathcal{P}_Y^m = \frac{1}{\mathbb{P}(C)} \sum_{Z \in \alpha_m} \mathbb{P}_C(C_Z)\mathcal{P}_{T_Z}, \tag{3.14}$$

where $\mathcal{P}_{T_Z}$ is as in (2.5).

lemma3.4 **Lemma 3.5** (cf. Lemmata V.2.1 and V.2.2 of [15])**.** *$\mathcal{P}_Y$ admits a fixed point $\varphi \in L^1(J, \lambda_J)$ that is bounded, locally Lipschitz and bounded away from zero.*

Proof. For each $m \in \mathbb{N}$ and $x \in J$,

$$\mathcal{P}_Y^m 1(x) = \frac{1}{\mathbb{P}(C)} \sum_{\substack{Z \in \alpha_m: \\ x \in T_Z(J_Z)}} \frac{\mathbb{P}_C(C_Z)}{|DT_Z(T_Z^{-1}x)|}. \tag{3.15}$$

Using the Mean Value Theorem, for all $m \in \mathbb{N}$ and $Z \in \alpha_m$ there exists a $\xi \in J_Z$ such that

q:mvt
$$\frac{\lambda\big(T_Z(J_Z)\big)}{\lambda(J_Z)} = |DT_Z(\xi)|. \tag{3.16}$$

Set $K_1 = \frac{\max\{K^{(\bar\rho)}, M^{(\bar\rho)}\}}{\mathbb{P}(C)\cdot\min\{\lambda(J_0),\lambda(J_1)\}}$, where $\bar\rho$ is as in (3.12) and (3.13). Since $DT_Z(\xi)$ and $DT_Z(y)$ have the same sign for any $y \in J_Z$, (3.16) together with (3.12) implies

q:bounded
$$\mathcal{P}_Y^m 1(x) \le \sum_{Z \in \alpha_m} \frac{\mathbb{P}_C(C_Z)}{\mathbb{P}(C)} \cdot K^{(\bar\rho)} \frac{\lambda(J_Z)}{\lambda(T_Z(J_Z))} \le K_1 \sum_{Z \in \alpha_m} \mathbb{P}_C \times \lambda_J(C_Z \times J_Z) = K_1. \tag{3.17}$$

Moreover, if for $A = [\mathbf{u}g^\kappa t] \times I \in \alpha$ we take $x, y \in I$, then for any $Z \in \alpha_m$ it holds that $x \in T_Z(J_Z)$ if and only if $y \in T_Z(J_Z)$. For such Z, let $x_Z, y_Z \in J_Z$ be such that $T_Z(x_Z) = x$ and $T_Z(y_Z) = y$. Then by (3.13)

q:equict
$$\begin{aligned} |\mathcal{P}_Y^m 1(x) - \mathcal{P}_Y^m 1(y)| &\le \sum_{\substack{Z \in \alpha_m: \\ x \in T_Z(J_Z)}} \frac{\mathbb{P}_C(C_Z)}{\mathbb{P}(C)} \left| \frac{1}{|DT_Z(x_Z)|} - \frac{1}{|DT_Z(y_Z)|} \right| \\ &\le \sum_{\substack{Z \in \alpha_m: \\ x \in T_Z(J_Z)}} \mathbb{P}_C(C_Z) \frac{1}{|DT_Z(x_Z)|} K_1 |T_Z(x_Z) - T_Z(y_Z)| \\ &= K_1 \mathcal{P}_Y^m 1(x) |x - y|. \end{aligned} \tag{3.18}$$

Together (3.17) and (3.18) imply that the sequence $\left(\frac{1}{m}\sum_{j=0}^{m-1} \mathcal{P}_Y^j 1\right)_m$ is uniformly bounded and equicontinuous on I for each $A = [\mathbf{u}g^\kappa t] \times I$. By Lemma 3.3(2) it follows that the same holds on J. Hence, by the Arzela-Ascoli Theorem there exists a subsequence

$$\left(\frac{1}{m_k} \sum_{j=0}^{m_k - 1} \mathcal{P}_Y^j 1 \right)_{m_k}$$

converging uniformly to a function $\varphi : J \to [0, \infty)$ satisfying $\varphi \le K_1$ and for each $A = [\mathbf{u}g^\kappa t] \times I \in \alpha$ and $x, y \in I$,

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$$|\varphi(x) - \varphi(y)| \le K_1 \varphi(x) |x - y|. \tag{3.19}$$

Hence, φ is bounded and by Lemma 3.3(2) it is clear that φ is locally Lipschitz, meaning that for every $x \in J$ there exists a neighbourhood $U \subseteq J$ of x such that φ restricted to U is Lipschitz continuous. It is readily checked that φ is a fixed point of $\mathcal{P}_Y$, so that $\mathbb{P}_C \times \nu$ with $\nu = \varphi\, d\lambda$ is an invariant probability measure for F_Y.

What is left is to verify that for each $A = [\mathbf{u}g^\kappa t] \times I \in \alpha$ the map φ is bounded from below on the interior of I. Suppose that there is such an $A = [\mathbf{u}g^\kappa t] \times I$ for which $\inf_{x \in I} \varphi(x) = 0$.

Then from (3.19) it follows that $\varphi(y) = 0$ for all $y \in I$, hence $\nu(I) = 0$. Either $I \subseteq J_0$ or $I \subseteq J_1$. If $I \subseteq J_0$, then for any set $A' = [\mathbf{v}g^\kappa t] \times I' \in \alpha$ with $T_\mathbf{v}(I') = J_0$ it holds that

$$\mathbb{P}_C \times \lambda_J(A' \cap F_Y^{-1}A) > 0$$

and, by the F_Y-invariance of $\mathbb{P}_C \times \nu$,

$$\mathbb{P}_C \times \nu(A' \cap F_Y^{-1}A) \leq \mathbb{P}_C \times \nu(F_Y^{-1}A) = \mathbb{P}_C \times \nu(A) = 0,$$

which together give $\inf_{x \in I'} \varphi(x) = 0$ and therefore, like before, $\nu(I') = 0$. There are sets $A' = [\mathbf{v}g^\kappa t] \times I'$ with $I' \subseteq J_1$ and $T_\mathbf{v}(I') = J_0$, so we can repeat the argument to show that also for any set $A'' = [\mathbf{v}g^\kappa t] \times I'' \in \alpha$ with $T_\mathbf{v}(I'') = J_1$ we have $\nu(I'') = 0$. So $\mathbb{P}_C \times \nu(A) = 0$ for all $A \in \alpha$. If $I \subseteq J_1$ we come to the same conclusion. This gives a contradiction, so φ is bounded from below on each interval I. □

It follows from Lemma 3.4 that $\mathbb{P}_C \times \nu$ with $\nu = \varphi d\lambda_J$ is a finite F_Y-invariant measure. To show that F_Y is ergodic with respect to $\mathbb{P} \times \lambda_J$ we need the following results, which states that the sets $\pi(A)$ for $A \in \alpha_m$ shrink uniformly to λ-null sets as $m \to \infty$.

:shrinking **Lemma 3.6.** $\lim_{m \to \infty} \sup\{\lambda_J(J_Z) : Z \in \alpha_m\} = 0.$

Proof. Set $\delta = \sup\{\lambda_J(J_Z) : Z \in \alpha\} < 1$. Fix an m and let $Z = \bigcap_{j=0}^{m-1} F_Y^{-j}([\mathbf{u}_j g^\kappa t] \times I_j) = C_Z \times J_Z \in \alpha_m$ as in (3.11). Set

$$\tilde{J}_Z = \bigcap_{j=0}^{m-2} T^{-1}_{\mathbf{u}_0\mathbf{u}_1\cdots\mathbf{u}_{j-1}}(I_j),$$

so that $J_Z = \tilde{J}_Z \cap T^{-1}_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(I_{m-1})$. Let J_i, $i \in \{0,1\}$, be such that $T_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(\tilde{J}_Z) = J_i$. It holds that $T_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(J_Z) = I_{m-1}$, so $\lambda(T_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(J_Z)) \leq \delta$ and thus

$$\lambda(T_{\mathbf{u}_0\mathbf{u}_1\cdots\mathbf{u}_{m-2}}(\tilde{J}_Z \setminus J_Z) \geq \lambda(J_i) - \delta.$$

Since $\tilde{J}_Z \setminus J_Z$ consists of at most two intervals, with (3.12) and (2.13) this gives

$$1 - \frac{\lambda_J(J_Z)}{\lambda_J(\tilde{J}_Z)} = \frac{\lambda_J(\tilde{J}_Z \setminus J_Z)}{\lambda_J(\tilde{J}_Z)} \geq \frac{1}{K^{(\bar{\rho})}} \frac{\lambda_J(T_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(\tilde{J}_Z \setminus J_Z)}{\lambda_J(T_{\mathbf{u}_0\cdots\mathbf{u}_{m-2}}(\tilde{J}_Z))} \geq \frac{1}{K^{(\bar{\rho})}} \frac{\lambda_J(J_i) - \delta}{\lambda_J(J_i)}.$$

Set $K_1 := \max\left\{1 - \frac{1}{K^{(\bar{\rho})}} \frac{\lambda_J(J_i)-\delta}{\lambda_J(J_i)} : i = 0,1\right\} \in (0,1)$. Then by repeating the same steps, we obtain

$$\lambda_J(J_Z) \leq K_1 \lambda_J(\tilde{J}_Z) \leq \cdots \leq K_1^m \lambda_J(I_0) < K_1^m,$$

which proves the lemma. □

Lemma 3.7. *F_Y is ergodic w.r.t. $\mathbb{P}_C \times \lambda_J$.*

Proof. Suppose $E \subseteq Y$ with $\mathbb{P}_C \times \lambda_J(E) > 0$ satisfies $F_Y^{-1}E = E \mod \mathbb{P}_C \times \lambda_J$. We show that $\mathbb{P}_C \times \lambda_J(E) = 1$. The Borel measure ρ on Y given by

$$\rho(V) = \int_V 1_E(\omega, x)\varphi(x) d\mathbb{P}_C(\omega) d\lambda_J(x)$$

for Borel sets V is F_Y-invariant. According to Lemma 2.2 and Lemma 2.1 this yields a stationary measure $\tilde{\mu}$ on $[0,1]$ that is absolutely continuous w.r.t. λ and satisfies $(\mathbb{P}\times\tilde{\mu})|_Y = \rho$. Let $L := \operatorname{supp}(\tilde{\mu}|_J)$ denote the support of the measure $\tilde{\mu}|_J$. Since ρ is a product measure, this gives $\operatorname{supp}(\rho) = C \times L$ and so by the definition of ρ we get $C \times L \subseteq E$ and $\rho(E\backslash(C \times L)) = 0$. Since φ is bounded away from zero, this yields

eqn3.17
$$E = C \times L \mod \mathbb{P}_C \times \lambda_J. \tag{3.20}$$

To obtain the result, it remains to show that $\lambda_J(J\backslash L) = 0$.

We have $C \times L = \bigcup_{Z\in\alpha_m} C_Z \times (J_Z \cap L)$ and $F_Y^{-m}(C \times L) = \bigcup_{Z\in\alpha_m} C_Z \times T_Z^{-1}L$. From the nonsingularity of F_Y w.r.t. $\mathbb{P}_C \times \lambda_J$ it follows that for each $m \in \mathbb{N}$,

eqn3.18
$$C \times L = E = F_Y^{-m}E = F_Y^{-m}(C \times L) \mod \mathbb{P}_C \times \lambda_J, \tag{3.21}$$

which yields

eqn3.19
$$J_Z \cap L = T_Z^{-1}L \mod \lambda_J, \quad \text{for each } Z \in \alpha_m. \tag{3.22}$$

Let $\varepsilon > 0$. Since $\lambda_J(L) > 0$, it follows from Lemma 3.6 and the Lebesgue Density Theorem that there are $i \in \{0,1\}$, $m_i \in \mathbb{N}$ and $Z_i \in \alpha_{m_i}$ such that

$$T_{Z_i}(J_{Z_i}) = J_i \quad \text{and} \quad \lambda_J(J_{Z_i} \cap L) \geq (1-\varepsilon)\lambda_J(J_{Z_i}).$$

By (3.22), $T_{Z_i}^{-1}(J_i \setminus L) = J_{Z_i} \setminus L \mod \lambda_J$. The Mean Value Theorem gives the existence of a $\xi \in J_{Z_i}$ such that

$$\frac{\lambda_J(T_{Z_i}(J_{Z_i}))}{\lambda_J(J_{Z_i})} = |DT_{Z_i}(\xi)|,$$

and from (3.12) it follows that

$$\lambda_J(T_{Z_i}(J_{Z_i} \setminus L)) = \int_{J_{Z_i}\setminus L} |DT_{Z_i}| d\lambda \leq K^{(\bar{\rho})}|DT_{Z_i}(\xi)|\lambda_J(J_{Z_i} \setminus L).$$

Hence,

eqn3.23
$$\frac{\lambda_J(J_i\backslash L)}{\lambda_J(J_i)} = \frac{\lambda_J(T_{Z_i}(J_{Z_i} \setminus L))}{\lambda_J(T_{Z_i}(J_{Z_i}))} \leq K^{(\bar{\rho})}\frac{\lambda_J(J_{Z_i}\backslash L)}{\lambda_J(J_{Z_i})} \leq K^{(\bar{\rho})}\varepsilon. \tag{3.23}$$

So, for each $\varepsilon > 0$ we can find an $i = i(\varepsilon)$ for which (3.23) holds. If for each $\varepsilon_0 > 0$ and each $i_0 \in \{0,1\}$ there exists an $\varepsilon \in (0,\varepsilon_0)$ such that $i(\varepsilon) = i_0$, we obtain from (3.23) that $\lambda_J(J\backslash L) = 0$. Otherwise, there exists $\varepsilon_0 > 0$ and $i_0 \in \{0,1\}$ such that $i(\varepsilon) = i_0$ for all $\varepsilon \in (0,\varepsilon_0)$. Without loss of generality, suppose that $i_0 = 0$. Then (3.23) gives $\lambda_J(J_0\backslash L) = 0$. By the equivalence of ν and λ_J and the fact that every good map has full branches it follows that

$$\mathbb{P}_C \times \nu\big((C \times J_0) \cap F_Y^{-1}(C \times J_1)\big) > 0. \tag{3.24}$$

Together with the Poincaré Recurrence Theorem this gives that

$$A = \{(\omega,x) \in C \times J_0 : F_Y^m(\omega,x) \in C \times J_1 \text{ for infinitely many } m \in \mathbb{N}\} \tag{3.25}$$

satisfies $\mathbb{P}_C \times \nu(A) > 0$, and therefore $\mathbb{P}_C \times \lambda_J(A) > 0$. Together with $\lambda_J(J_0\backslash L) = 0$ it follows from the Lebesgue Density Theorem that there exists a Lebesgue point $x \in \pi(A)\cap L$

of $1_{\pi(A)\cap L}$. Since $x \in \pi(A)$, for infinitely many $m \in \mathbb{N}$ there exists $Z_m \in \alpha_m$ such that $x \in J_{Z_m}$ and $T_{Z_m}(J_{Z_m}) = J_1$. This again together with Lemma 3.6 yields that for each $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $Z \in \alpha_m$ such that

$$T_Z(J_Z) = J_1 \quad \text{and} \quad \lambda_J(J_Z \cap L) \geq (1-\varepsilon)\lambda_J(J_Z).$$

Similar as before, this gives $\lambda_J(J_1 \backslash L) = 0$, so $\lambda_J(J \backslash L) = 0$. □

subsec:3.3

3.3. **The proof of Theorem 1.1.**

In the previous paragraphs we collected all the ingredients necessary to prove Theorem 1.1.

Proof of Theorem 1.1. (1) According to Lemma 2.2 and Lemma 2.1 the finite F_Y-invariant measure $\mathbb{P}_C \times \nu$ yields a $(\mathbb{P} \times \lambda)$-absolutely continuous F-invariant measure which is of the form $\mathbb{P} \times \mu$ with μ stationary and absolutely continuous w.r.t. λ, and satisfying $\mu|_J = \nu$. Lemma 3.2 gives that μ is σ-finite. Furthermore, combining Lemma 3.2 with Maharam's Recurrence Theorem gives that F is conservative with respect to $\mathbb{P} \times \mu$ as well. From the ergodicity of F_Y with respect to $\mathbb{P}_C \times \lambda_J$ it then follows by Lemma 3.2 that F is ergodic with respect to $\mathbb{P} \times \lambda$ and thus also with respect to $\mathbb{P} \times \mu$ by Lemma 3.5. We conclude that μ is the unique (up to scalar multiplication) stationary measure that is absolutely continuous w.r.t. λ. For more details see e.g. [1, Paragraph 1.5].

(2) For the density $\psi := \frac{d\mu}{d\lambda}$ it holds that $\psi|_J = \varphi$. Since we can take κ in the definition of J as large as we want, ψ is locally Lipschitz on $(0, c)$ and $(c, 1)$. Moreover, it is a fixed point of the Perron-Frobenius operator from (2.6) and thus for all $x \in [0, 1]$,

q:psipf

$$\psi(x) = \mathcal{P}_F^{\kappa}\psi(x) \geq p_g^{\kappa} \frac{\varphi(T_g^{-\kappa}x)}{|DT_g^{\kappa}(T_g^{-\kappa}x)|}. \tag{3.26}$$

From Lemma 3.5 we conclude that ψ is bounded from below by some constant $C > 0$. It remains to show that ψ is not in L^q for any $q > 1$. To see this, fix a $b \in \Sigma_B$. Since ψ is bounded from below by $C > 0$, we have for all $k \in \mathbb{Z}_{\geq 0}$ and $x \in [0, 1]$ that

$$\psi(x) = \mathcal{P}_F^{k+1}\psi(x) \geq C \cdot p_g p_b^k \sum_{y \in (T_g T_b^k)^{-1}\{x\}} \frac{1}{|D(T_g T_b^k)(y)|}. \tag{3.27}$$

Let $\ell_b, M_b, r_g, M_g, K_g$ be as in (B4) and (G4). From (B4), (G4) and Lemma 2.4 we get

q:q1

$$\begin{aligned} |D(T_g T_b^k)(y)| &= |DT_g(T_b^k(y))| \prod_{i=1}^{k} |DT_b(T_b^{k-i}(y))| \\ &\leq M_g |T_b^k(y) - c|^{r_g - 1} \prod_{i=0}^{k-1} (M_b |T_b^i(y) - c|^{\ell_b - 1}) \\ &\leq M_g M_b^k (\tilde{M}|y - c|)^{\ell_b^k (r_g - 1)} \prod_{i=0}^{k-1} (\tilde{M}|y - c|)^{\ell_b^i(\ell_b - 1)} \\ &= K_1 |y - c|^{\ell_b^k r_g - 1}, \end{aligned} \tag{3.28}$$

for the positive constant $K_1 = M_g M_b^k \tilde{M}^{\ell_b^k r_g - 1}$. On the other hand, from (G4) we obtain for any $y \in (T_g T_b^k)^{-1}\{x\}$ as in the proof of Lemma 2.4 that

$$|x - T_g(c)| = |T_g T_b^k(y) - T_g(c)| \geq \frac{K_g}{r_g} |T_b^k(y) - c|^{r_g}$$

and then Lemma 2.4 yields

q:q2

$$|x - T_g(c)| \geq K_2 |y - c|^{\ell_b^k r_g} \tag{3.29}$$

for the positive constant $K_2 = \frac{K_g}{r_g} \tilde{K}^{\ell_b r_g}$. Now for any $q > 1$ we can choose $k \in \mathbb{Z}_{\geq 0}$ large enough so that $\tau := (1 - \ell_b^{-k} r_g^{-1}) q \geq 1$. Combining (3.26), (3.28) and (3.29) we obtain

$$\begin{aligned} \psi^q(x) &\geq \Big(\frac{C p_g p_b^k}{K_1}\Big)^q \Big(\sum_{y \in (T_g T_b^k)^{-1}\{x\}} |y - c|^{1 - \ell_b^k r_g} \Big)^q \\ &\geq K_3 |x - T_g(c)|^{-\tau} \end{aligned}$$

for a positive constant K_3. This gives the result. □

remark3.1

Remark 3.1. *The result from Theorem 1.1 still holds if we allow the critical order ℓ_b from (B4) to be equal to 1 for some b, as long as $\ell_{\max} > 1$. To see this, note that in the proof of Theorem 1.1 condition (B4) only plays a role in proving that $\frac{d\mu_{\mathbf{p}}}{d\lambda} \notin L^q$ for any $q > 1$. Here we refer to Lemma 2.4 and the constants $\tilde{K}$ and $\tilde{M}$, which are not well defined if $\ell_{\min} = 1$. In (3.28) however, we use the estimates from Lemma 2.4 only for one arbitrary fixed $b \in \Sigma_B$. By the same reasoning as in the proof of Lemma 2.4 it follows that*

$$\bigg(\Big(\frac{K_b}{\ell_b}\Big)^{\frac{1}{\ell_b - 1}} |x - c|\bigg)^{\ell_b^n} \leq |T_b^n(x) - c| \leq \bigg(\Big(\frac{M_b}{\ell_b}\Big)^{\frac{1}{\ell_b - 1}} |x - c|\bigg)^{\ell_b^n}. \tag{3.30}$$

for any $b \in \Sigma_B$ with $\ell_b > 1$. Hence, if there exists at least one $b \in \Sigma_B$ with $\ell_b > 1$, then we can replace the bounds obtained from Lemma 2.4 in (3.28) and (3.29) by constants $K_1 = M_g M_b^k (\frac{K_b}{\ell_b})^{(\ell_b^k r_g - 1)/(\ell_b - 1)}$ and $K_2 = \frac{K_g}{r_g} (\frac{M_b}{\ell_b})^{\ell_b r_g / (\ell_b - 1)}$ and obtain the same result. In case $\ell_{\max} = 1$, then most parts from Theorem 1.1 still remain valid with the exception that then we can only say that $\frac{d\mu_{\mathbf{p}}}{d\lambda} \notin L^q$ if $q \geq \frac{r_{\max}}{r_{\max} - 1}$. This follows from the above reasoning by taking $k = 0$ in the definition of τ in the proof of Theorem 1.1 and by noting that $\tau = (1 - r_{\max}^{-1}) q \geq 1$ if $q \geq \frac{r_{\max}}{r_{\max} - 1}$.

:in-finite

4. Estimates on the acs measure

In this section we prove Theorem 1.2. Recall the definition of θ from Theorem 1.2:

$$\theta = \sum_{b \in \Sigma_B} p_b \ell_b.$$

subsec4.1

4.1. **The case $\theta \geq 1$.** To prove one direction of Theorem 1.2, namely that the unique acs measure μ from Theorem 1.1 is infinite if $\theta \geq 1$, we introduce another induced transformation.

prop3.2

Proposition 4.1. *Suppose $\theta \geq 1$. Then the unique acs measure μ from Theorem 1.1 is infinite.*

Proof. Fix a $b \in \Sigma_B$. Recall the definitions of $\tilde{M}$ from Lemma 2.4 and δ from in and below the proof of Lemma 2.4, and set $\gamma = \min\{\delta, \frac{1}{2}\tilde{M}^{-1}\}$. Let $a \in [c-\gamma, c)$. Then there exists a $\xi \in (a, c)$ such that $T_b(a) > \xi$ and $T_b^2(a) > \xi$. Take $[bb] \times (a, \xi)$ as the inducing domain and let

$$\kappa(\omega, x) = \inf\{k \in \mathbb{N} : F^k(\omega, x) \in [bb] \times (a, \xi)\} \tag{4.1}$$

be the first return time to $[bb] \times (a, \xi)$ under F. If $\mathbb{P} \times \mu([bb] \times (a, \xi)) = \infty$, there is nothing left to prove. If not, then we compute $\int_{[bb]\times(a,\xi)} \kappa \, d\mathbb{P} \times \mu$ and use Kac's Formula from Lemma 2.3 to prove the result.

So, assume that $\mathbb{P} \times \mu([bb] \times (a, \xi)) < \infty$. The conditions that $T_b(a) > \xi$ and $T_b^2(a) > \xi$ together with the fact that any bad map has c as a fixed point and is strictly monotone on the intervals $[0, c]$ and $[c, 1]$, guarantee that for each $n \in \mathbb{N}$ and $\omega \in \Sigma_B^{\mathbb{N}} \cap [bb]$ we get

eqn3.30

$$T_\omega^n((a, \xi)) \cap (a, \xi) = \emptyset. \tag{4.2}$$

For any $\omega \in [bb]$ and $x \in (a, \xi)$ it follows by (4.2) and (2.16) that $T_\omega^n(x)$ can only return to (a, ξ) after at least one application of a good map. Assume that $\omega \in [bb]$ is of the form

$$\omega = (b, b, \omega_3, \omega_4, \ldots, \omega_n, g, \omega_{n+2}, \ldots),$$

with $n \geq 2$, $\omega_i \in \Sigma_B$ for $3 \leq i \leq n$, $g \in \Sigma_g$, and $x \in (a, \xi)$. Then $\kappa(\omega, x) \geq n+1$. Lemma 2.4 yields that

eqn3.31

$$|T_\omega^n(x) - c| \leq (\tilde{M}\gamma)^{\ell_{\omega_1} \cdots \ell_{\omega_n}} < 2^{-\ell_{\omega_1} \cdots \ell_{\omega_n}}. \tag{4.3}$$

From (G4) and (4.3) we obtain that

eqn3.32

$$|T_g T_\omega^n(x) - T_g(c)| = \left| \int_c^{T_\omega^n(x)} DT_g(y) \, dy \right| \leq \frac{M_g}{r_g} |T_\omega^n(x) - c|^{r_g} < \frac{M_g}{r_g} \cdot 2^{-\ell_{\omega_1} \cdots \ell_{\omega_n} r_g}. \tag{4.4}$$

Set

$$\zeta = \sup\{|DT_j(x)| \, : \, j \in \Sigma, \, x \in [0, 1]\}. \tag{4.5}$$

Then $\zeta > 1$ by (G5), (B5). Assume $\kappa(\omega, x) = m+n$ for some $m \geq 1$. Then $T_\omega^{m+n}(x) \in (a, \xi)$ so that by (G3),

$$|T_\omega^{m+n}(x) - T_g(c)| \geq \min\{a, 1-\xi\}. \tag{4.6}$$

Because of (4.4) this implies

$$\zeta^{m-1} \frac{M_g}{r_g} \cdot 2^{-\ell_{\omega_1} \cdots \ell_{\omega_n} r_g} \geq \min\{a, 1-\xi\}. \tag{4.7}$$

Solving for m yields

$$m \geq K_1 + K_2 \ell_{\omega_1} \cdots \ell_{\omega_n} \tag{4.8}$$

for constants $K_1 = \left(1 + \log\left(\frac{\min\{a,1-\xi\} r_g}{M_g}\right)\right) / \log \zeta \in \mathbb{R}$ and $K_2 = \log(2^{r_g}) / \log \zeta > 0$. Note that K_1, K_2 are independent of ω, x, m and n.

We obtain that for any $g \in \Sigma_G$,

$$\int_{[bb]\times(a,\xi)} \kappa \, d\mathbb{P} \times \mu \geq \sum_{n \in \mathbb{N}_{\geq 2}} \sum_{\omega_3,\ldots,\omega_n \in \Sigma_B} \mathbb{P}([bb\omega_3 \cdots \omega_n g]) \mu((a,\xi)) \Big(n + K_1 + K_2 \ell_b^2 \prod_{i=3}^{n} \ell_{\omega_i}\Big).$$

Since

$$\sum_{n \in \mathbb{N}_{\geq 2}} \sum_{\omega_3,\ldots,\omega_n \in \Sigma_B} \mathbb{P}([\omega_3 \cdots \omega_n]) \prod_{i=3}^{n} \ell_{\omega_i} = 1 + \sum_{n \in \mathbb{N}} \theta^n = \infty,$$

we get $\int_{[bb]\times(a,\xi)} \kappa \, d\mathbb{P} \times \mu = \infty$ and from Lemma 2.3 we now conclude that μ is infinite. □

sec3.3

4.2. **The case $\theta < 1$.** For the other direction of Theorem 1.2, assume $\theta < 1$. We first obtain a stationary probability measure $\tilde{\mu}$ for $(\mathcal{T}, \mathbf{p})$ using a standard Krylov-Bogolyubov type argument. For this, let $\mathcal{M}$ denote the set of all finite Borel measures on $[0,1]$, and define the operator $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{P}\nu = \sum_{j \in \Sigma} p_j \nu \circ T_j^{-1}, \qquad \nu \in \mathcal{M}, \tag{4.9}$$

where $\nu \circ T_j^{-1}$ denotes the pushforward measure of ν under T_j. Then $\mathcal{P}$ is a *Markov-Feller* operator (see e.g. [24]) with dual operator U on the space $B([0,1])$ of all bounded Borel measurable functions given by[1] $Uf = \sum_{j \in \Sigma} p_j f \circ T_j$ for $f \in B([0,1])$. As before, let λ denote the Lebesgue measure on $[0,1]$, and set $\lambda_n = \mathcal{P}^n \lambda$ for each $n \geq 0$. Furthermore, for each $n \in \mathbb{N}$ define the Cesáro mean $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k$. Since the space of probability measures on $[0,1]$ equipped with the weak topology is sequentially compact, there exists a subsequence $(\mu_{n_k})_{k \in \mathbb{N}}$ of $(\mu_n)_{n \in \mathbb{N}}$ that converges weakly to a probability measure $\tilde{\mu}$ on $[0,1]$. Using that a Markov-Feller operator is weakly continuous, it then follows from a standard argument that $\mathcal{P}\tilde{\mu} = \tilde{\mu}$, that is, $\tilde{\mu}$ is a stationary probability $\tilde{\mu}$ for $(\mathcal{T}, \mathbf{p})$. The next theorem will lead to the estimate (1.5) from Theorem 1.2. For any $\mathbf{b} = b_1 \cdots b_k \in \Sigma_B^k$, $k \geq 0$, recall that we abbreviate $p_{\mathbf{b}} = \prod_{i=1}^{k} p_{b_i}$ and also let $\ell_{\mathbf{b}} = \prod_{i=1}^{k} \ell_{b_i}$ where we use $p_{\mathbf{b}} = 1 = \ell_{\mathbf{b}}$ in case $k = 0$.

[1]By definition of a Markov-Feller operator, the space of bounded *continuous* functions is required to be invariant under the dual operator U. If there is a $g \in \Sigma_G$ for which T_g is discontinuous (namely at c), we then first identify $[0,1]$ with the unit circle S^1 so that T_g can be viewed as a continuous map on S^1. With the same identification any acs measure on S^1 then gives an acs measure on $[0,1]$.

thrm3.1 **Theorem 4.1.** *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all Borel sets $A \subseteq [0,1]$ we have*

eqn3.40
$$\lambda_n(A) \le C \cdot \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{\infty} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot \lambda(A)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}. \tag{4.10}$$

Before we prove this theorem, we first show how it gives Theorem 1.2.

Proof of Theorem 1.2. The first part of the statement follows from Proposition 4.1. For the second part, assume that $\theta < 1$ and that Theorem 4.1 holds. Let $A \subseteq [0,1]$. Using the regularity of λ, for any $\delta > 0$ there exists an open set $G \subseteq [0,1]$ such that $A \subseteq G$ and $\lambda(G) \le \lambda(A) + \delta$. Using that $(\mu_{n_k})_{k \in \mathbb{N}}$ converges weakly to $\tilde{\mu}$, we obtain from the Portmanteau Theorem together with Theorem 4.1 that

eqn3.41
$$\begin{aligned} \tilde{\mu}(A) \le \tilde{\mu}(G) &\le \liminf_k \mu_{n_k}(G) \\ &\le C \cdot \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{\infty} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot (\lambda(A) + \delta)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}. \end{aligned} \tag{4.11}$$

Since $\theta < 1$, the sum is bounded and with the Dominated Convergence Theorem we can take the limit as $\delta \to 0$ to obtain

eqn3.42
$$\tilde{\mu}(A) \le C \cdot \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{\infty} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot \lambda(A)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}. \tag{4.12}$$

This proves that $\tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. It follows that the probability measure $\tilde{\mu}$ is equal to the unique acs measure $\mu_{\mathbf{p}}$ from Theorem 1.1. The estimate (1.5) follows directly from (4.12). ☐

It remains to give the proof of Theorem 4.1. We shall do this in a number of steps.

prop0.6 **Proposition 4.2.** *There exists a constant $K_1 > 0$ such that for all $n \in \mathbb{N}$, all $\mathbf{u} \in \Sigma^n$ and all Borel sets $A \subseteq [0,1]$ with $0 < 3\lambda(A) < \frac{1}{2}\min\{c, 1-c\}$ we have*

$$\lambda(T_{\mathbf{u}}^{-1}A) \le K_1 \big(\lambda(T_{\mathbf{u}}^{-1}[0,3\eta)) + \lambda(T_{\mathbf{u}}^{-1}(c-3\eta, c+3\eta)) + \lambda(T_{\mathbf{u}}^{-1}(1-3\eta,1]) \big),$$

where $\eta = \lambda(A)$.

Proof. Let $n \in \mathbb{N}$, $\mathbf{u} \in \Sigma^n$ and a Borel set $A \subseteq [0,1]$ with $0 < 3\lambda(A) < \frac{1}{2}\min\{c, 1-c\} < 1$ be given and write $\eta = \lambda(A)$. The map $T_{\mathbf{u}}$ has non-positive Schwarzian derivative on any of its intervals of monotonicity (see (2.9)) and the image of any such interval is $[0,c]$, $[c,1]$ or $[0,1]$. Set $A_1 = (\eta, c-\eta)$ and $A_2 = (2\eta, c-2\eta)$. Let I be a connected component of $T_{\mathbf{u}}^{-1}A_1$, and set $f = T_{\mathbf{u}}|_I$ and $I^* = f^{-1}A_2$. The Minimum Principle yields

eq19
$$|Df(x)| \ge \min_{z \in \partial I^*} |Df(z)|, \qquad \text{for all } x \in I^*. \tag{4.13}$$

Suppose the minimal value is attained at $f^{-1}(2\eta)$ and set $A_3 = (2\eta, 3\eta)$ and $J = f^{-1}A_3$. By the condition on the size of A it follows from the Koebe Principle that

eq20
$$K^{(\eta)}|Df(f^{-1}(2\eta))| \geq |Df(x)|, \qquad \text{for all } x \in J. \tag{4.14}$$

Combining (4.13) and (4.14) gives

$$\begin{aligned}\lambda(f^{-1}(A \cap A_2)) &= \int_{A\cap A_2} \frac{1}{|Df(f^{-1}y)|}\, d\lambda(y) \leq \lambda(A) \cdot \frac{1}{|Df(f^{-1}(2\eta))|} \\ &\leq K^{(\eta)} \int_{A_3} \frac{1}{|Df(f^{-1}y)|}\, d\lambda(y) = K^{(\eta)} \lambda(f^{-1}(A_3)).\end{aligned}$$

We conclude that

eq21
$$\lambda\big(T_{\mathbf{u}}^{-1}\big(A \cap (2\eta, c-2\eta)\big)\big) \leq K^{(\eta)} \lambda\big(T_{\mathbf{u}}^{-1}(2\eta, 3\eta)\big). \tag{4.15}$$

In case $\min_{z\in\partial I^*} |Df(z)| = f^{-1}(c-2\eta)$, a similar reasoning yields

$$\lambda\big(T_{\mathbf{u}}^{-1}\big(A \cap (2\eta, c-2\eta)\big)\big) \leq K^{(\eta)} \lambda\big(T_{\mathbf{u}}^{-1}(c-3\eta, c-2\eta)\big). \tag{4.16}$$

Furthermore, a similar reasoning can be done for the interval $[c, 1]$ to conclude that

$$\lambda\big(T_{\mathbf{u}}^{-1}\big(A \cap (c+2\eta, 1-2\eta)\big)\big) \leq K^{(\eta)} \Big(\lambda\big(T_{\mathbf{u}}^{-1}(c+2\eta, c+3\eta)\big) + \lambda\big(T_{\mathbf{u}}^{-1}(1-3\eta, 1-2\eta)\big)\Big).$$

Hence, setting $K_1 = \max\{K^{(\eta)}, 1\}$ gives the desired result. □

Proposition 4.2 shows that to get the desired estimate from Theorem 4.1 it suffices to consider small intervals on the left and right of $[0, 1]$ and around c, i.e., sets of the form

$$I_c(\varepsilon) := (c-\varepsilon, c+\varepsilon) \quad \text{and} \quad I_0(\varepsilon) := [0, \varepsilon) \cup (1-\varepsilon, 1]$$

for $\varepsilon > 0$. We first focus on estimating the measure of the intervals $I_c(\varepsilon)$.

lemma0.8 **Lemma 4.1.** *There exists a constant $K_2 \geq 1$ such that for all $n \in \mathbb{N}$, $\mathbf{u} \in \Sigma^{n-1} \times \Sigma_G$ and all $\varepsilon > 0$ we have*

$$\lambda(T_{\mathbf{u}}^{-1} I_c(\varepsilon)) \leq K_2 \varepsilon. \tag{4.17}$$

Proof. Let $n \in \mathbb{N}$ and $\mathbf{u} \in \Sigma^{n-1} \times \Sigma_G$. Let $\varepsilon > 0$. Suppose that $\varepsilon \geq \frac{1}{4}\min\{c, 1-c\}$. Then

eq3.53
$$\lambda(T_{\mathbf{u}}^{-1} I_c(\varepsilon)) \leq 1 \leq \frac{4\varepsilon}{\min\{c, 1-c\}}. \tag{4.18}$$

Now suppose $\varepsilon < \frac{1}{4}\min\{c, 1-c\}$. Again the map $T_{\mathbf{u}}$ has non-positive Schwarzian derivative on the interior of any of its intervals of monotonicity and since $u_n \in \Sigma_G$ the image of any such interval is $[0, 1]$. Use $\mathcal{I}$ to denote the collection of connected components of $T_{\mathbf{u}}^{-1} I_c(\varepsilon)$. Let $A \in \mathcal{I}$ and write $J = J_A$ and $I = I_A$ for the intervals that satisfy $A \subseteq J$, $A \subseteq I$ and

$$\begin{aligned} T_{\mathbf{u}}(J) &= \Big[c - \frac{1}{2}\min\{c, 1-c\}, c + \frac{1}{2}\min\{c, 1-c\}\Big], \\ T_{\mathbf{u}}(I) &= \Big[c - \frac{3}{4}\min\{c, 1-c\}, c + \frac{3}{4}\min\{c, 1-c\}\Big]. \end{aligned}$$

Also, write $f = T_{\mathbf{u}}|_I$. Since f has non-positive Schwarzian derivative, it follows from (2.13) that

$$\frac{\lambda(A)}{\lambda(J)} \le K^{(\frac{1}{4})} \frac{\lambda(f(A))}{\lambda(f(J))} = K^{(\frac{1}{4})} \frac{2\varepsilon}{\min\{c, 1-c\}}. \tag{4.19}$$

We conclude that

eq3.57a
$$\lambda(T_{\mathbf{u}}^{-1} I_c(\varepsilon)) = \sum_{A \in \mathcal{I}} \lambda(A) \le K^{(\frac{1}{4})} \frac{2\varepsilon}{\min\{c,1-c\}} \sum_{A \in \mathcal{I}} \lambda(J_A) \le K^{(\frac{1}{4})} \frac{2\varepsilon}{\min\{c,1-c\}}. \tag{4.20}$$

Defining $K_2 = \frac{2\max\{2, K^{(\frac{1}{4})}\}}{\min\{c,1-c\}}$, the desired result now follows from (4.18) and (4.20). □

To find $\lambda_n\big(I_c(\varepsilon)\big)$, first note that from Lemma 2.4 it follows that for all $\varepsilon > 0$, $n \in \mathbb{N}$, $\mathbf{b} \in \Sigma_B^n$,

eq3.57
$$T_{\mathbf{u}}^{-1}\big(I_c(\varepsilon)\big) \subseteq I_c\big(\tilde{K}^{-1} \varepsilon^{\ell_{\mathbf{u}}^{-1}}\big). \tag{4.21}$$

By splitting Σ^n according to the final block of bad indices, we can then write using (4.21) and Lemma 4.1 that

$$\begin{aligned}
\lambda_n\big(I_c(\varepsilon)\big) &= \sum_{k=0}^{n-1} \sum_{\mathbf{v} \in \Sigma^{n-k-1}} \sum_{g \in \Sigma_G} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{v}g\mathbf{b}} \lambda\big(T_{\mathbf{v}g\mathbf{b}}^{-1} I_c(\varepsilon)\big) + \sum_{\mathbf{b} \in \Sigma_B^n} p_{\mathbf{b}} \lambda\big(T_{\mathbf{b}}^{-1} I_c(\varepsilon)\big) \\
&\le \sum_{k=0}^{n-1} \sum_{\mathbf{v} \in \Sigma^{n-k-1}} \sum_{g \in \Sigma_G} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{v}g\mathbf{b}} \lambda\big(T_{\mathbf{v}g}^{-1} I_c(\tilde{K}^{-1} \varepsilon^{\ell_{\mathbf{b}}^{-1}})\big) + \sum_{\mathbf{b} \in \Sigma_B^n} p_{\mathbf{b}} \lambda\big(I_c(\tilde{K}^{-1} \varepsilon^{\ell_{\mathbf{b}}^{-1}})\big) \\
&\le \sum_{k=0}^{n-1} \sum_{g \in \Sigma_G} \sum_{\mathbf{b} \in \Sigma_B^k} p_g p_{\mathbf{b}} K_2 \tilde{K}^{-1} \varepsilon^{\ell_{\mathbf{b}}^{-1}} + \sum_{\mathbf{b} \in \Sigma_B^n} p_{\mathbf{b}} 2\tilde{K}^{-1} \varepsilon^{\ell_{\mathbf{b}}^{-1}}.
\end{aligned}$$

Taking $K_3 = \max\big\{K_2, 2\big(\sum_{g \in \Sigma_G} p_g\big)^{-1}\big\} \cdot \tilde{K}^{-1} \ge 1$ then gives

:measureIm
$$\lambda_n\big(I_c(\varepsilon)\big) \le K_3 \sum_{g \in \Sigma_G} \sum_{k=0}^{n} \sum_{\mathbf{b} \in \Sigma_B^k} p_g p_{\mathbf{b}} \varepsilon^{\ell_{\mathbf{b}}^{-1}}. \tag{4.22}$$

We now focus on $I_0(\varepsilon) = [0, \varepsilon) \cup (1-\varepsilon, 1]$. Fix an $0 < \varepsilon_0 < \frac{1}{2}\min\{c, 1-c\}$ and a $t > 1$ that satisfy

eq31
$$|DT_j(x)| > t, \qquad \text{for all } x \in I_0(\varepsilon_0) \text{ and each } j \in \Sigma. \tag{4.23}$$

Such ε_0 and t exist because of (G5) and (B5). From (G4) it follows that for each $0 < \varepsilon < \varepsilon_0$ and $g \in \Sigma_G$,

$$|T_g(x) - T_g(c)| = \left| \int_c^x DT_g(y) dy \right| \ge \frac{K_g}{r_g} \cdot |x - c|^{r_g}.$$

Set $K_4 = \max\{(K_g^{-1} r_g)^{r_g^{-1}} : g \in \Sigma_G\} \ge 1$. Then (G1) and (G3) imply that

eq3.60
$$T_g^{-1} I_0(\varepsilon) \subseteq I_0(\varepsilon t^{-1}) \cup I_c(K_4 \varepsilon^{r_g^{-1}}). \tag{4.24}$$

Furthermore, from (B1) and (B3) it follows that for each $\varepsilon \in (0, \varepsilon_0)$ and $b \in \Sigma_B$,

eq3.61 (4.25)
$$T_b^{-1} I_0(\varepsilon) \subseteq I_0(\varepsilon t^{-1}).$$

Write each $\mathbf{u} \in \Sigma^n$ as

eq:4.30 (4.26)
$$\mathbf{u} = \mathbf{b}_1 \mathbf{g}_1 \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}$$

for some $\tilde{s} \in \{1, \dots, n\}$, where for each i we have $\mathbf{b}_i = b_{i,1} \cdots b_{i,k_i} \in \Sigma_B^{k_i}$ and $\mathbf{g}_i = g_{i,1} \cdots g_{i,m_i} \in \Sigma_G^{m_i}$ for some $k_1, m_{\tilde{s}} \in \mathbb{Z}_{\geq 0}$ and $k_2, \dots, k_{\tilde{s}}, m_1, \dots, m_{\tilde{s}-1} \in \mathbb{N}$. Define

$$s = \begin{cases} \tilde{s}, & \text{if } m_{\tilde{s}} \geq 1, \\ \tilde{s} - 1, & \text{if } m_{\tilde{s}} = 0. \end{cases}$$

Moreover, we introduce notation to indicate the length of the tails of the block $\mathbf{u}$:

$$\begin{aligned} d_i &= |\mathbf{b}_i \mathbf{g}_i \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}|, && i \in \{1, \dots, \tilde{s}\}, \\ q_{i,j} &= |g_{i,j+1} \cdots g_{i,m_i} \mathbf{b}_{i+1} \mathbf{g}_{i+1} \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}|, && i \in \{1, \dots, \tilde{s}\},\ j \in \{0, \dots, m_i\}. \end{aligned}$$

If necessary to avoid confusion, we write $s(\mathbf{u})$, $k_i(\mathbf{u})$, etcetera to emphasize the dependence on $\mathbf{u}$.

l:c5 **Lemma 4.2.** *There exists a constant $K_5 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, $n \in \mathbb{N}$ and $\mathbf{u} = \mathbf{b}_1 \mathbf{g}_1 \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}} \in \Sigma^n$,*

$$\begin{aligned} T_{\mathbf{u}}^{-1} I_0(\varepsilon) \subseteq I_0(\varepsilon t^{-d_1}) &\cup \bigcup_{i=1}^{s} T^{-1}_{\mathbf{b}_1 \mathbf{g}_1 \cdots \mathbf{b}_{i-1} \mathbf{g}_{i-1}} I_c\Big(K_5 (\varepsilon t^{-q_{i,1}})^{\ell_{\mathbf{b}_i}^{-1} r_{g_{i,1}}^{-1}}\Big) \\ &\cup \bigcup_{i=1}^{s} \bigcup_{j=2}^{m_i} T^{-1}_{\mathbf{b}_1 \mathbf{g}_1 \cdots \mathbf{b}_{i-1} \mathbf{g}_{i-1} \mathbf{b}_i g_{i,1} \cdots g_{i,j-1}} I_c(K_5 (\varepsilon t^{-q_{i,j}})^{r_{g_{i,j}}^{-1}}). \end{aligned}$$

Proof. We prove the statement by an induction argument for $\tilde{s}$. Let $\mathbf{u}$ be a word with symbols in Σ, and write $\mathbf{u} = \mathbf{b}_1 \mathbf{g}_1 \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}$ for its decomposition as in (4.26). First suppose that $\tilde{s} = 1$. If $m_1 = 0$, then the statement immediately follows from repeated application of (4.25). If $m_1 \geq 1$, then repeated application of (4.24) gives

(4.27)
$$T_{\mathbf{g}_1}^{-1} I_0(\varepsilon) \subseteq I_0(\varepsilon t^{-q_{1,0}}) \cup I_c\Big(K_4 (\varepsilon t^{-q_{1,1}})^{r_{g_{1,1}}^{-1}}\Big) \cup \bigcup_{j=2}^{m_1} T^{-1}_{g_{1,1} \cdots g_{1,j-1}} I_c\Big(K_4 (\varepsilon t^{-q_{1,j}})^{r_{g_{1,j}}^{-1}}\Big).$$

By setting $K_5 = \tilde{K}^{-1} K_4$, applying (4.21) and (4.25) then yields

$$T_{\mathbf{b}_1 \mathbf{g}_1}^{-1} I_0(\varepsilon) \subseteq I_0(\varepsilon t^{-d_1}) \cup I_c\Big(K_5 (\varepsilon t^{-q_{1,1}})^{\ell_{\mathbf{b}_1}^{-1} r_{g_{1,1}}^{-1}}\Big) \cup \bigcup_{j=2}^{m_1} T^{-1}_{\mathbf{b}_1 g_{1,1} \cdots g_{1,j-1}} I_c\Big(K_5 (\varepsilon t^{-q_{1,j}})^{r_{g_{1,j}}^{-1}}\Big).$$

Note that this is true for the case that $k_1 = 0$ as well. This proves the statement if $\tilde{s} = 1$. Now suppose $\tilde{s}(\mathbf{u}) > 1$ and suppose that the statement holds for all words $\mathbf{v}$ with

$\tilde{s}(\mathbf{v}) = \tilde{s}(\mathbf{u}) - 1$. In particular, the statement then holds for the word $\mathbf{b}_2\mathbf{g}_2 \cdots \mathbf{b}_{\tilde{s}}\mathbf{g}_{\tilde{s}}$. Note that $m_1 \geq 1$. Again, by repeated application of (4.24) it follows that

eq37
$$T_{\mathbf{g}_1}^{-1} I_0(\varepsilon t^{-d_2}) \subseteq I_0(\varepsilon t^{-q_{1,0}}) \cup I_c\Big(K_4(\varepsilon t^{-q_{1,1}})^{r_{g_{1,1}}^{-1}}\Big) \cup \bigcup_{j=2}^{m_1} T_{g_{1,1}\cdots g_{1,j-1}}^{-1} I_c\Big(K_4(\varepsilon t^{-q_{1,j}})^{r_{g_{1,j}}^{-1}}\Big). \tag{4.28}$$

Furthermore, applying (4.21) and (4.25) then yields

$$T_{\mathbf{b}_1\mathbf{g}_1}^{-1} I_0(\varepsilon t^{-d_2}) \subseteq I_0(\varepsilon t^{-d_1}) \cup I_c\Big(K_5(\varepsilon t^{-q_{1,1}})^{\ell_{\mathbf{b}_1}^{-1} r_{g_{1,1}}^{-1}}\Big) \cup \bigcup_{j=2}^{m_1} T_{\mathbf{b}_1 g_{1,1}\cdots g_{1,j-1}}^{-1} I_c\Big(K_5(\varepsilon t^{-q_{1,j}})^{r_{g_{1,j}}^{-1}}\Big).$$

This together with the statement being true for the word $\mathbf{b}_2\mathbf{g}_2 \cdots \mathbf{b}_{\tilde{s}}\mathbf{g}_{\tilde{s}}$ yields the statement for $\mathbf{u}$. □

Combining Lemma 4.1 and Lemma 4.2 gives

$$\lambda(T_{\mathbf{u}}^{-1} I_0(\varepsilon)) \leq 2\varepsilon t^{-d_1} + \sum_{i=1}^{s} K_2 K_5 (\varepsilon t^{-q_{i,1}})^{\ell_{\mathbf{b}_i}^{-1} r_{g_{i,1}}^{-1}} + \sum_{i=1}^{s}\sum_{j=2}^{m_i} K_2 K_5 (\varepsilon t^{-q_{i,j}})^{r_{g_{i,j}}^{-1}}.$$

Let $r_{\max} = \max\{r_g : g \in \Sigma_G\}$ and set $\alpha := t^{1/r_{\max}} > 1$. Then

$$\sum_{i=1}^{s}\sum_{j=2}^{m_i} \alpha^{-q_{i,j}} \leq \sum_{\ell=0}^{\infty} \alpha^{-\ell} = \frac{1}{1-1/\alpha},$$

so that

q:c2c5
$$\begin{aligned} \lambda(T_{\mathbf{u}}^{-1} I_0(\varepsilon)) &\leq 2\varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{i=1}^{s}\sum_{j=2}^{m_i} \varepsilon^{1/r_{\max}} \alpha^{-q_{i,j}} + \sum_{i=1}^{s} K_2 K_5 (\varepsilon t^{-q_{i,1}})^{\ell_{\mathbf{b}_i}^{-1} r_{g_{i,1}}^{-1}} \\ &\leq \left(2 + \frac{K_2 K_5}{1-1/\alpha}\right)\varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{i=1}^{s} (\varepsilon t^{-q_{i,1}})^{\ell_{\mathbf{b}_i}^{-1} r_{g_{i,1}}^{-1}}. \end{aligned} \tag{4.29}$$

prop15
Proposition 4.3. *There exists a constant $K_6 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and $n \in \mathbb{N}$,*

$$\lambda_n(I_0(\varepsilon)) \leq K_6 \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{n-1} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot \varepsilon^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}.$$

Proof. Let $n \in \mathbb{N}$. Then with (4.29) we obtain

eqnr1
$$\begin{aligned} \lambda_n(I_0(\varepsilon)) &= \sum_{\mathbf{u} \in \Sigma^n} p_{\mathbf{u}} \lambda\big(T_{\mathbf{u}}^{-1}(I_0(\varepsilon))\big) \\ &\leq \left(2 + \frac{K_2 K_5}{1-1/\alpha}\right)\varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{\mathbf{u} \in \Sigma^n} p_{\mathbf{u}} \sum_{i=1}^{s(\mathbf{u})} \big(\varepsilon t^{-q_{i,1}(\mathbf{u})}\big)^{\ell_{\mathbf{b}_i(\mathbf{u})}^{-1} r_{g_{i,1}(\mathbf{u})}^{-1}} \\ &= \left(2 + \frac{K_2 K_5}{1-1/\alpha}\right)\varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{i=1}^{\tau} \sum_{\mathbf{u} \in \Sigma^n} 1_{\{1,\ldots,s(\mathbf{u})\}}(i) p_{\mathbf{u}} \big(\varepsilon t^{-q_{i,1}(\mathbf{u})}\big)^{\ell_{\mathbf{b}_i(\mathbf{u})}^{-1} r_{g_{i,1}(\mathbf{u})}^{-1}}, \end{aligned} \tag{4.30}$$

where we defined $\tau = \lfloor \frac{n+1}{2} \rfloor$ which is the largest value $s(\mathbf{u})$ can take. Let us consider the second term in (4.30). First of all, note that a word $\mathbf{u} \in \Sigma^n$ satisfies $s(\mathbf{u}) \geq 1$ if and only if $m_1(\mathbf{u}) \geq 1$. Therefore,

$$\{\mathbf{u} \in \Sigma^n : s(\mathbf{u}) \geq 1\} = \bigcup_{k=0}^{n-1} \Sigma_B^k \times \Sigma_G \times \Sigma^{n-k-1}.$$

Hence, defining the function χ on $\{0, \dots, n-1\}^2$ by

defchi (4.31)
$$\chi(k,q) = \sum_{\mathbf{b} \in \Sigma_B^k} \sum_{g \in \Sigma_G} p_{\mathbf{b}} p_g \big(\varepsilon t^{-q}\big)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}, \qquad (k,q) \in \{0, \dots, n-1\}^2.$$

we can rewrite and bound the term with $i = 1$ in (4.30) as follows:

eqnr2 (4.32)
$$\begin{aligned} \sum_{\mathbf{u} \in \Sigma^n} 1_{\{1,\dots,s(\mathbf{u})\}}(1) p_{\mathbf{u}} \big(\varepsilon t^{-q_{1,1}(\mathbf{u})}\big)^{\ell_{\mathbf{b}_i(\mathbf{u})}^{-1} r_{g_{1,1}(\mathbf{u})}^{-1}} &= \sum_{k=0}^{n-1} \sum_{\mathbf{v} \in \Sigma^{n-k-1}} p_{\mathbf{v}} \chi(k, n-k-1) \\ &\leq \varepsilon^{1/r_{\max}} + \sum_{k=1}^{n-1} \chi(k, n-k-1). \end{aligned}$$

Secondly, note that for each $i \in \{2, \dots, \tau\}$ a word $\mathbf{u} \in \Sigma^n$ satisfies $s(\mathbf{u}) \geq i$ if and only if $m_{i-1}(\mathbf{u}), k_i(\mathbf{u}), m_i(\mathbf{u}) \geq 1$. For each $k \in \{1, \dots, n-1\}$ and $q \in \{0, \dots, n-k-2\}$ and $i \in \{2, \dots, \tau\}$ we define

(4.33)
$$A_{i,k,q} = \{\mathbf{v} \in \Sigma^{n-k-q-1} : \tilde{s}(\mathbf{v}) = i-1, v_{n-k-q-1} \in \Sigma_G\}.$$

The set $A_{i,k,q}$ contains all words of length $n-k-q-1$ that can precede the word $\mathbf{b}_i \mathbf{g}_i \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}$ with $|\mathbf{b}_i| = k$ and $|g_{i,2} \cdots g_{i,m_i} \mathbf{b}_{i+1} \mathbf{g}_{i+1} \cdots \mathbf{b}_{\tilde{s}} \mathbf{g}_{\tilde{s}}| = q$. So

$$\{\mathbf{u} \in \Sigma^n : s(\mathbf{u}) \geq i\} = \bigcup_{k=1}^{n-1} \bigcup_{q=0}^{n-k-2} A_{i,k,q} \times \Sigma_B^k \times \Sigma_G \times \Sigma^q, \qquad i \in \{2, \dots, \tau\}.$$

Hence, using (4.31) we can rewrite and bound the sum in (4.30) that runs from $i = 2$ to τ as follows:

eqnr3 (4.34)
$$\begin{aligned} \sum_{i=2}^{\tau} \sum_{\mathbf{u} \in \Sigma^n} 1_{\{1,\dots,s(\mathbf{u})\}}(i) p_{\mathbf{u}} \big(\varepsilon t^{-q_{i,1}(\mathbf{u})}\big)^{\ell_{\mathbf{b}_i(\mathbf{u})}^{-1} r_{g_{i,1}(\mathbf{u})}^{-1}} &= \sum_{i=2}^{\tau} \sum_{k=1}^{n-1} \sum_{q=0}^{n-k-2} \sum_{\mathbf{v}_1 \in A_{i,k,q}} \sum_{\mathbf{v}_2 \in \Sigma^{q-1}} p_{\mathbf{v}_1} p_{\mathbf{v}_2} \chi(k,q) \\ &= \sum_{k=1}^{n-1} \sum_{q=0}^{n-k-2} \chi(k,q) \sum_{i=2}^{\tau} \sum_{\mathbf{v}_1 \in A_{i,k,q}} \sum_{\mathbf{v}_2 \in \Sigma^{q-1}} p_{\mathbf{v}_1} p_{\mathbf{v}_2} \\ &\leq \sum_{k=1}^{n-1} \sum_{q=0}^{n-k-2} \chi(k,q). \end{aligned}$$

Here the last step follows from the fact that

$$\sum_{i=2}^{\tau} \sum_{\mathbf{v}_1 \in A_{i,k,q}} p_{\mathbf{v}_1} \leq \sum_{\mathbf{v} \in \Sigma^{n-k-q-2}} \sum_{g \in \Sigma_G} p_{\mathbf{v}} p_g \leq 1. \tag{4.35}$$

Combining (4.32) and (4.34) gives

eqnr4
$$\sum_{i=1}^{\tau} \sum_{\mathbf{u} \in \Sigma^n} 1_{\{1,\dots,s(\mathbf{u})\}}(i) p_{\mathbf{u}} \big(\varepsilon t^{-q_{i,1}(\mathbf{u})}\big)^{\ell_{\mathbf{b}_i(\mathbf{u})}^{-1} r_{g_{i,1}(\mathbf{u})}^{-1}} \leq \varepsilon^{1/r_{\max}} + \sum_{k=1}^{n-1} \sum_{q=0}^{n-k-1} \chi(k,q). \tag{4.36}$$

Furthermore, for each $\mathbf{b} \in \Sigma_B^k$ and $g \in \Sigma_G$ we have again by setting $r_{\max} = \max\{r_j : j \in \Sigma_G\}$ and $\alpha = t^{1/r_{\max}}$ that

eqnr5
$$\sum_{q=0}^{n-k-1} (t^{-q})^{\ell_{\mathbf{b}}^{-1} r_g^{-1}} \leq \sum_{q=0}^{n-k-1} \big(\alpha^{-\ell_{\mathbf{b}}^{-1}}\big)^q \leq \frac{1}{1-\alpha^{-\ell_{\mathbf{b}}^{-1}}} \leq \frac{\alpha \ell_{\mathbf{b}}^{-1}}{\alpha^{\ell_{\mathbf{b}}^{-1}} - 1} \ell_{\mathbf{b}} \leq \frac{\alpha}{\log(\alpha)} \ell_{\mathbf{b}}, \tag{4.37}$$

where the last step follows from the fact that $f(x) = \frac{x}{\alpha^x - 1}$ is a decreasing function and $\lim_{x \downarrow 0} f(x) = \frac{1}{\log \alpha}$. Hence, combining (4.30), (4.36) and (4.37) gives

$$\begin{aligned}
\lambda_n(I_0(\varepsilon)) &\leq \left(2 + \frac{K_2 K_5}{1 - 1/\alpha} + K_2 K_5\right) \varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{k=1}^{n-1} \sum_{q=0}^{n-k-1} \sum_{\mathbf{b} \in \Sigma_B^k} \sum_{g \in \Sigma_G} p_{\mathbf{b}} p_g \big(\varepsilon t^{-q}\big)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}} \\
&\leq \left(2 + K_2 K_5 \frac{2\alpha - 1}{\alpha - 1}\right) \varepsilon^{1/r_{\max}} + K_2 K_5 \sum_{k=1}^{n-1} \sum_{\mathbf{b} \in \Sigma_B^k} \sum_{g \in \Sigma_G} p_{\mathbf{b}} p_g \varepsilon^{\ell_{\mathbf{b}}^{-1} r_g^{-1}} \frac{\alpha \ell_{\mathbf{b}}}{\log(\alpha)} \\
&\leq K_6 \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{n-1} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \varepsilon^{\ell_{\mathbf{b}}^{-1} r_g^{-1}},
\end{aligned}$$

where $K_6 = \frac{1}{\min\{p_g : g \in \Sigma_G\}} \big(2 + K_2 K_5 \frac{2\alpha - 1}{\alpha - 1}\big) + \frac{K_2 K_5 \alpha}{\log \alpha}$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $A \subseteq [0,1]$ be a Borel set. First suppose that $\lambda(A) \geq \frac{\varepsilon_0}{3}$. Then there exists a constant $C > 0$ large enough such that

$$\lambda_n(A) \leq 1 \leq C \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{\infty} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot \lambda(A)^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}. \tag{4.38}$$

Now suppose that $\lambda(A) < \frac{\varepsilon_0}{3}$ and set $\varepsilon = 3\lambda(A)$. It follows from Proposition 4.2 that for all $n \in \mathbb{N}$ and all $\mathbf{u} \in \Sigma^n$ we have

$$\lambda\big(T_{\mathbf{u}}^{-1} A\big) \leq K_1\big(\lambda(T_{\mathbf{u}}^{-1} I_0(\varepsilon)) + \lambda(T_{\mathbf{u}}^{-1} I_c(\varepsilon))\big).$$

Together with (4.22) and Proposition 4.3 this yields for all $n \in \mathbb{N}$ that

$$\lambda_n(A) \le K_1 \cdot (K_3 + K_6) \sum_{g \in \Sigma_G} p_g \sum_{k=0}^{\infty} \sum_{\mathbf{b} \in \Sigma_B^k} p_{\mathbf{b}} \ell_{\mathbf{b}} \cdot \varepsilon^{\ell_{\mathbf{b}}^{-1} r_g^{-1}}.$$

This gives the result. □

sec3c4

5. Further results and final remarks

5.1. **Stability in p.** In this section we prove Corollary 1.1 on the stability of the absolutely continuous stationary measures in the parameter $\mathbf{p}$. The proof of Corollary 1.1 consists of two steps. Firstly we show that any weak limit point of μ_n is a stationary measure, i.e., satisfies (2.3), and secondly that any weak limit point of μ_n is absolutely continuous with respect to the Lebesgue measure. The corollary then follows from the uniqueness of absolutely continuous stationary measures given by Theorem 1.1.

Proof of Corollary 1.1. For each $n \ge 0$ let $\mathbf{p}_n = (p_{n,j}) \in \mathbb{R}^N$ be a positive probability vector with $\sum_{j \in \Sigma_B} p_{n,j} \ell_j < 1$ and suppose that $\lim_{n\to\infty} \mathbf{p}_n = \mathbf{p}_0$ in $\mathbb{R}^N$. Let $\tilde{\mu}$ be a weak limit point of μ_n. Again, note that such a $\tilde{\mu}$ exists because the space of probability measures on $[0,1]$ equipped with the weak topology is sequentially compact. After passing to a subsequence we have for any continuous function $\varphi : [0,1] \to \mathbb{R}$ that

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\tilde{\mu}.$$

Moreover, by the stationarity of the measures μ_n it follows that for each $n \ge 1$,

$$\int_{[0,1]} \varphi \, d\mu_n = \sum_{j=1}^{N} p_{n,j} \int_{[0,1]} \varphi \circ T_j \, d\mu_n.$$

To prove that $\tilde{\mu}$ is stationary for $\mathbf{p}_0$, it is sufficient to show that for each j,

eq:intphii

$$\lim_{n\to\infty} p_{n,j} \int_{[0,1]} \varphi \circ T_j \, d\mu_n = p_{0,j} \int_{[0,1]} \varphi \circ T_j \, d\tilde{\mu}. \tag{5.1}$$

If $j \in \Sigma_B$ this is obvious, since then $\varphi \circ T_j$ is continuous. For $j \in \Sigma_G$ the map $\varphi \circ T_j$ might have a discontinuity at c. In this case, we let φ_δ be the continuous function given by $\varphi_\delta(x) = \varphi \circ T_j(x)$ for $x \in I \setminus (c-\delta, c+\delta)$ and φ_δ is linear otherwise. Then we have

$$\lim_{n\to\infty} \left| p_{n,j} \int_{[0,1]} \varphi_\delta \, d\mu_n - p_{0,j} \int_{[0,1]} \varphi_\delta \, d\mu_0 \right| = 0,$$

by the weak convergence and since $p_{n,j} \to p_{0,j}$ as $n \to \infty$. Also, we have

$$\left| p_{n,j} \int_{[0,1]} \varphi \circ T_j d\mu_n - p_{n,j} \int_{[0,1]} \varphi_\delta d\mu_n \right| \le C \mu_n([c-\delta, c+\delta]) \to 0 \text{ as } \delta \to 0,$$

where the convergence is uniform in n because of (1.5). The last two relations imply (5.1).

To show that $\tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure λ we proceed as in the proof of Theorem 1.2. First note that since $\mathbf{p}_n \to \mathbf{p}_0$, it also holds that

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \sum_{j\in\Sigma_B} p_{n,j}\ell_j = \theta_0 < 1,$$

so that $\tilde{\theta} := \sup_n \theta_n < 1$. Let $A \subseteq [0,1]$ be a Borel set. By Theorem 1.1 every μ_n satisfies (1.5), so that

$$\mu_n(A) \le C_n \sum_{k=0}^{\infty} \tilde{\theta}^k \lambda(A)^{\ell_{\max}^{-k} r_{\max}^{-1}},$$

where the constant C_n depends on $\big(\sum_{g\in\Sigma_G} p_{n,g}\big)^{-1}$ and $(\min\{p_{n,g} : g \in \Sigma_G\})^{-1}$ (and properties of the good and bad maps themselves that are not linked to the probabilities). Since each $\mathbf{p}_n$, $n \ge 0$, is a positive probability vector and $\mathbf{p}_n \to \mathbf{p}_0$, both these quantities can be bounded from above and $\tilde{C} := \sup_n C_n < \infty$. From the weak convergence of μ_n to $\tilde{\mu}$ we obtain as in (4.12) using the Portmanteau Theorem that

$$\tilde{\mu}(A) \le \tilde{C} \sum_{k=0}^{\infty} \tilde{\theta}^k \lambda(A)^{\ell_{\max}^{-k} r_{\max}^{-1}}.$$

Hence, $\tilde{\mu} \ll \lambda$. By Theorem 1.1 we know that μ_0 is the unique acs probability measure for F and $\mathbf{p}_0$. So, $\tilde{\mu} = \mu_0$. □

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5.2. **The non-superattracting case.** With some modifications the results from Theorem 1.1 and Theorem 1.2 can be extended to the class $\mathfrak{B}^1 \supseteq \mathfrak{B}$ of bad maps of which critical order ℓ_b in (B4) is allowed to be equal to 1. We will list the modified statements and the necessary modifications to the proofs here. Note that for each $T \in \mathfrak{B}^1 \backslash \mathfrak{B}$, we have $DT(c) \neq 0$, and due to the minimal principle, $|DT(c)| < 1$. So we consider $T_1, \dots, T_N \in \mathfrak{G} \cup \mathfrak{B}^1$ with $\Sigma_B^1 = \{1 \le j \le N : T_j \in \mathfrak{B}^1\}$ and Σ_G, Σ_B as before and such that $\Sigma_G, \Sigma_B^1 \backslash \Sigma_B \neq \emptyset$. Furthermore, we write again $\Sigma = \{1, \dots, N\} = \Sigma_G \cup \Sigma_B^1$.

MAIN3 **Theorem 5.1.** *Let $\{T_j : j \in \Sigma\}$ be as above and $\mathbf{p} = (p_j)_{j\in\Sigma}$ a positive probability vector.*

(1) There exists a unique (up to scalar multiplication) stationary σ-finite measure $\mu_{\mathbf{p}}$ for F that is absolutely continuous with respect to the one-dimensional Lebesgue measure λ. This measure is ergodic and the density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is bounded away from zero and is locally Lipschitz on $(0,c)$ and $(c,1)$.

(2) Suppose $\ell_{\max} > 1$.

(i) The measure $\mu_{\mathbf{p}}$ is finite if and only if $\theta = \sum_{b\in\Sigma_B^1} p_b \ell_b < 1$. In this case, there exists constants $C > 0$ and $\hat{\theta} \in (\theta, 1)$ such that

eq:5.2

$$\mu_{\mathbf{p}}(A) \le C \cdot \sum_{k=0}^{\infty} \hat{\theta}^k \lambda(A)^{\ell_{\max}^{-k} r_{\max}^{-1}} \tag{5.2}$$

for any Borel set $A \subseteq [0,1]$, where $r_{\max} = \max\{r_g : g \in \Sigma_G\}$ and $\ell_{\max} = \max\{\ell_b : b \in \Sigma_B\}$.

(ii) The density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is not in L^q for any $q > 1$.

(3) Suppose $\ell_{\max} = 1$.

(i) The measure $\mu_{\mathbf{p}}$ is finite, and for each $\boldsymbol{\eta} = (\eta_b)_{b\in\Sigma_B^1}$ such that $\eta_b > 1$ for each $b \in \Sigma_B^1$ and $\hat{\theta}(\boldsymbol{\eta}) = \sum_{b\in\Sigma_B^1} p_b\eta_b < 1$ there exists a constant $C(\boldsymbol{\eta}) > 0$ such that

eq:5.3
$$\mu_{\mathbf{p}}(A) \le C(\boldsymbol{\eta}) \cdot \sum_{k=0}^{\infty} \hat{\theta}(\boldsymbol{\eta})^k \lambda(A)^{\eta_{\max}^{-k} r_{\max}^{-1}} \tag{5.3}$$

for any Borel set $A \subseteq [0,1]$, where $\eta_{\max} = \max\{\eta_b : b \in \Sigma_B^1\}$. If $\sum_{b\in\Sigma_B^1} \frac{p_b}{|DT_b(c)|} < 1$, so if the bad maps are expanding on average at the point c, then we can get the estimate

eq:5.4
$$\mu_{\mathbf{p}}(A) \le C \cdot \lambda(A)^{r_{\max}^{-1}} \tag{5.4}$$

for some constant $C > 0$ and any Borel set $A \subseteq [0,1]$.

(ii) If $r_{\max} > 1$, then $\frac{d\mu_{\mathbf{p}}}{d\lambda} \notin L^q$ for any $q \ge \frac{r_{\max}}{r_{\max}-1}$. If, moreover, $\sum_{b\in\Sigma_B^1} \frac{p_b}{|DT_b(c)|} < 1$, then $\frac{d\mu_{\mathbf{p}}}{d\lambda} \in L^q$ for all $1 \le q < \frac{r_{\max}}{r_{\max}-1}$.

(iii) If $r_{\max} = 1$ and $\sum_{b\in\Sigma_B^1} \frac{p_b}{|DT_b(c)|} < 1$, then $\frac{d\mu_{\mathbf{p}}}{d\lambda} \in L^\infty$.

The main issue we need to deal with in order to get Theorem 5.1 is adapting Lemma 2.4, i.e., finding suitable bounds for $|T_\omega^n(x) - c|$, since the constants $\tilde{K}$ and $\tilde{M}$ from Lemma 2.4 are not well defined in case $\ell_{\min} = 1$. This is done in the next two lemmata. For the upper bound of $|T_\omega^n(x) - c|$ we assume $\ell_{\max} > 1$ since we only need it for the proof of part (2)(i).

lemma5.1
Lemma 5.1. *Let $\{T_j : j \in \Sigma\}$ be as above. Suppose $\ell_{\max} > 1$. There are constants $\hat{M} > 1$ and $\delta > 0$ such that for all $n \in \mathbb{N}$, $\omega \in (\Sigma_B^1)^{\mathbb{N}}$ and $x \in [c-\delta, c+\delta]$ we have*

eq5.2
$$|T_\omega^n(x) - c| \le \Big(\hat{M}|x-c|\Big)^{\ell_{\omega_1}\cdots\ell_{\omega_n}}. \tag{5.5}$$

Proof. Similar as in the proof of Lemma 2.4 it follows that there exists an $M > 1$ such that for any $b \in \Sigma_B$ and $x \in [0,1]$ we have

eq5.7
$$|T_b(x) - c| \le M|x-c|^{\ell_b}. \tag{5.6}$$

Furthermore, there exists a $\delta > 0$ such that $|DT_b(x)| < 1$ for all $x \in [c-\delta, c+\delta]$ and $b \in \Sigma_B^1$. This implies

eq5.8
$$|T_b(x) - c| < |x-c| \tag{5.7}$$

for all $x \in [c-\delta, c+\delta]$ and $b \in \Sigma_B^1$. Note that $\Sigma_B \neq \emptyset$ because $\ell_{\max} > 1$. We set $\upsilon = \min\{\ell_b : b \in \Sigma_B\} > 1$ and $\hat{M} = M^{\frac{1}{\upsilon-1}}$. For each $n \in \mathbb{N}$ and $\omega \in (\Sigma_B^1)^{\mathbb{N}}$, write

$$m(n,\omega) = \#\{1 \le \omega_i \le n \,:\, \ell_{\omega_i} > 1\}. \tag{5.8}$$

The statement follows by showing that for all $n \in \mathbb{N}$, $\omega \in (\Sigma_B^1)^{\mathbb{N}}$ and $x \in [c-\delta, c+\delta]$ we have

eqn5.9
$$|T_\omega^n(x) - c| \le \Big(M^{(1-\upsilon^{-m(n,\omega)})/(\upsilon-1)}|x-c|\Big)^{\ell_{\omega_1}\cdots\ell_{\omega_n}}. \tag{5.9}$$

We prove (5.9) by induction. From (5.6) and (5.7) it follows that (5.9) holds for $n = 1$. Now suppose (5.9) holds for some $n \in \mathbb{N}$. Let $\omega \in (\Sigma_B^1)^{\mathbb{N}}$ and $y \in [c-\delta, c+\delta]$. If $\ell_{\omega_{n+1}} = 1$, then the desired result follows by applying (5.7) with $j = \omega_{n+1}$ and $x = T_\omega^n(y)$. Suppose $\ell_{\omega_{n+1}} > 1$. Then, using (5.6),

$$\begin{aligned}|T_\omega^{n+1}(y) - c| &\leq M|T_\omega^n(y) - c|^{\ell_{\omega_{n+1}}} \\ &\leq \Big(M^{(1-\upsilon^{-m(n,\omega)})/(\upsilon-1)+\upsilon^{-m(n+1,\omega)}}|y-c|\Big)^{\ell_{\omega_1}\cdots\ell_{\omega_{n+1}}}.\end{aligned}$$

Using that

$$\upsilon^{-m(n+1,\omega)} = \frac{\upsilon^{-m(n,\omega)} - \upsilon^{-m(n+1,\omega)}}{\upsilon - 1}, \tag{5.10}$$

the desired result follows. □

lemma5.2 **Lemma 5.2.** *Let $\{T_j : j \in \Sigma\}$ be as above. Let $\boldsymbol{\eta} = (\eta_b)_{b\in\Sigma_B^1}$ be a vector such that $\eta_b > 1$ for each $b \in \Sigma_B^1$. Set $\hat{\eta}_b = \max\{\eta_b, \ell_b\}$ for each $b \in \Sigma_B^1$. Then there exists a constant $\hat{K}(\boldsymbol{\eta}) \in (0,1)$ such that for all $n \in \mathbb{N}$, $\omega \in (\Sigma_B^1)^{\mathbb{N}}$ and $x \in [0,1]$ we have*

$$\Big(\hat{K}(\boldsymbol{\eta})|x-c|\Big)^{\hat{\eta}_{\omega_1}\cdots\hat{\eta}_{\omega_n}} \leq |T_\omega^n(x) - c|. \tag{5.11}$$

Proof. Note from (B4) that for each $b \in \Sigma_B^1$ we have

$$K_b|x-c|^{\hat{\eta}_b - 1} \leq K_b|x-c|^{\ell_b-1} \leq |DT_b(x)|.$$

The result now follows in the same way as in the proof of Lemma 2.4 by setting $\hat{\eta}_{\min} = \min\{\hat{\eta}_b : b \in \Sigma_B^1\}$, $\hat{\eta}_{\max} = \max\{\hat{\eta}_b : b \in \Sigma_B^1\}$ and $\hat{K}(\boldsymbol{\eta}) = \big(\frac{\min\{K_b : b\in\Sigma_B^1\}}{\hat{\eta}_{\max}}\big)^{\frac{1}{\hat{\eta}_{\min}-1}}$. □

Proof of Theorem 5.1. Firstly, note that (1), (2)(ii) and the first part of (3)(ii) immediately follow from Remark 3.1. Moreover, as in [15, Section 5.4] it can be shown that (5.4) implies that $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is in L^q if $r_{\max} > 1$ and $1 \leq q < \frac{r_{\max}}{r_{\max}-1}$, giving the remainder of 3(ii). It is immediate that (5.4) implies that $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is in L^∞ if $r_{\max} = 1$, so (3)(iii) holds. Hence, it remains to prove (2i) and (3i).

Suppose $\theta = \sum_{b\in\Sigma_B^1} p_b\ell_b \geq 1$, which means that $\ell_{\max} > 1$. The proof that in this case $\mu_{\mathbf{p}}$ is infinite follows by the same reasoning as in Subsection 4.1 by now taking $\gamma = \min\{\delta, \frac{1}{2}\hat{M}^{-1}\}$ with δ and $\hat{M}$ as in the proof of Lemma 5.1. Now suppose $\theta < 1$. Then there exists a vector $\boldsymbol{\eta} = (\eta_b)_{b\in\Sigma_B^1}$ such that $\eta_b > 1$ for each $b \in \Sigma_B^1$ and $\hat{\theta}(\boldsymbol{\eta}) = \sum_{b\in^1_B} p_b\hat{\eta}_b < 1$ with again $\hat{\eta}_b = \max\{\eta_b, \ell_b\}$. Applying Lemma 5.2 yields that for all $\varepsilon > 0$, $n \in \mathbb{N}$, $\mathbf{b} \in (\Sigma_B^1)^n$,

eq5.12
$$T_{\mathbf{b}}^{-1}\big(I_c(\varepsilon)\big) \subseteq I_c\big(\hat{K}(\boldsymbol{\eta})^{-1}\varepsilon^{\hat{\eta}_{\mathbf{b}}^{-1}}\big), \tag{5.12}$$

where we used the notation $\hat{\eta}_{\mathbf{b}} = \hat{\eta}_{b_1}\cdots\hat{\eta}_{b_n}$ for a word $\mathbf{b} = b_1\cdots b_n$. Following the line of reasoning in Subsection 4.2 with (5.12) instead of (4.21), we obtain that there exists a constant $C(\boldsymbol{\eta}) > 0$ such that

eq:5.13
$$\mu_{\mathbf{p}}(A) \le C(\boldsymbol{\eta}) \cdot \sum_{k=0}^{\infty} \hat{\theta}(\boldsymbol{\eta})^k \lambda(A)^{\hat{\eta}_{\max}^{-k} r_{\max}^{-1}} \tag{5.13}$$

for any Borel set $A \subseteq [0,1]$. In case $\ell_{\max} > 1$ we can choose $\boldsymbol{\eta}$ to satisfy $\hat{\eta}_{\max} = \ell_{\max}$, which yields (2)(i). In case $\ell_{\max} = 1$, then $\hat{\eta}_{\max} = \eta_{\max}$, so this together with (5.13) yields the first part of (3)(i).

Finally, for the second part of (3)(i), suppose $\ell_{\max} = 1$ and $\Lambda = \sum_{b \in \Sigma_B^1} \frac{p_b}{|DT_b(c)|} < 1$. Setting $K_{\mathbf{b}} = |DT_{\mathbf{b}}(c)|$ for each $\mathbf{b} \in (\Sigma_B^1)^n$ and $n \in \mathbb{N}$, note that for all $\varepsilon > 0$, $n \in \mathbb{N}$, $\mathbf{b} \in (\Sigma_B^1)^n$,

eq5.14
$$T_{\mathbf{b}}^{-1}\big(I_c(\varepsilon)\big) \subseteq I_c\big(K_{\mathbf{b}}^{-1}\varepsilon\big). \tag{5.14}$$

By using (5.14) instead of (4.21), letting $\tilde{p}_{\mathbf{b}} = K_{\mathbf{b}}^{-1} p_{\mathbf{b}}$ play the role of $p_{\mathbf{b}}$ in the reasoning of Subsection 4.2 and noting that $\Lambda^k = \sum_{\mathbf{b} \in (\Sigma_B^1)^k} \tilde{p}_{\mathbf{b}}$, we arrive similarly as for Theorem 4.1 to the conclusion that there exists a constant $\tilde{C} > 0$ such that for all $n \in \mathbb{N}$ and all Borel sets $A \subseteq [0,1]$,

eqn5.15
$$\lambda_n(A) \le \tilde{C} \cdot \sum_{g \in \Sigma_G} p_g \Big(\sum_{k=0}^{\infty} \Lambda^k \Big) \lambda(A)^{r_g^{-1}}. \tag{5.15}$$

This proves the remaining part of (3)(i). □

5.3. **Final remarks.** The results from Theorem 5.1 contain one possible extension of our main results to another set of conditions (G1)–(G5), (B1)–(B5). One could consider various other changes to these conditions. A condition that plays a fundamental role in the proofs of Theorem 1.1 and Theorem 1.2 is the fact that the critical point is mapped to a point that is a common repelling fixed point for all maps T_j. If this condition is relaxed, for instance by assuming that the branches of one of the good maps are not full, then the critical values of the random system are not just $0, c, 1$ but contain all the values of all possible postcritical orbits of c. For this reason, an invariant density (if it exists) clearly cannot be locally Lipschitz on $(0,c)$ and $(c,1)$. Furthermore, Proposition 4.2 and all subsequent arguments fail, since it is not sufficient to restrict to neighbourhoods around only 0, c and 1. One might try to solve this issue by requiring that the postcritical orbits 'gain enough expansion' as was done in for instance [28] for deterministic maps. An analogous condition for random systems, however, would become much stronger since it would have to hold for all possible random orbits of c. Another problem one needs to face when relaxing the surjectivity condition is that the argument using Kac's Lemma might fail, because in that case there exist words $\mathbf{u}$ with symbols in Σ and neighborhoods U of c such that $T_{\mathbf{u}}(x)$ is bounded away from zero and one uniformly in $x \in U$.

The dynamical behaviour of the system is governed by the interplay between the super-exponential convergence at c and the exponential divergence from 0 and 1. In this article we fixed the exponential divergence away from 0 and 1 and the two regimes $\theta < 1$ and

$\theta \geq 1$ in Theorem 1.2 only refer to the convergence at c: For smaller θ orbits are less attracted to c. It would be interesting to see under what other conditions on the rates of convergence to c and divergence from 0 and 1 the system admits an acs measure. Could one for example take exponential convergence to c and polynomial divergence from 0 and 1? Or could one replace the conditions (G5) and (B5) stating that all good and bad maps are expanding at 0 and 1 by the condition that the random system is expanding on average at a sufficiently large neighbourhood of 0 and 1?

There are also some additional questions that our main results raise. It would be interesting for example to study further statistical properties of the random system such as mixing properties and if possible mixing rates in case the acs measure is finite. It is not clear a priori if the behaviour of the good maps dominates the statistical properties of the random system, since trajectories spend long periods of time near the points 0, c and 1. In this respect the dynamics resembles that of the Manneville-Pomeau maps, and mixing rates might be polynomial rather than exponential. A way to approach this problem is by estimating the measures $\mathbb{P} \times \lambda(\{\varphi_Y > n\})$, where φ_Y is the first return time to Y defined in Section 3 as they give information on the rates of decay of correlations. To obtain the desired decay rates it is sufficient to obtain estimates for $\mathbb{P} \times \lambda(\{\varphi_Y = k\})$ for all $k > n$. Recall that every returning set $\{\varphi_Y = k\}$ is of the form $C_k \times J(C_k)$, where $C_k \subset \Sigma^{\mathbb{N}}$ is a cylinder set and $J \subset I$ is an interval with return time k, which depends only on C_k. Obtaining effective estimates on individual intervals J by directly looking at pre-images of Y under the skew product system does not seem very feasible at the moment, since cylinders can contain a positive proportion of bad maps. An alternative approach could be a combinatorial construction as in [3] or [13], where a two step induction process is introduced. To perform a similar construction we have to find a suitable way to define the binding period or the slow recurrence to the critical set, which takes into account the existence of bad maps.

Finally, in Theorems 1.1 and 5.1 we have seen that the regularity of the density $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ depends on whether or not there is a bad map for which c is superattracting: If $\ell_{\max} > 1$, then $\frac{d\mu_{\mathbf{p}}}{d\lambda}$ is not in L^q for any $q > 1$, whereas if $\ell_{\max} = 1$ and the bad maps are expanding on average at c, i.e. $\sum_{b \in \Sigma_B^1} \frac{p_b}{|DT_b(c)|} < 1$, then in the case that $r_{\max} > 1$ we have $\frac{d\mu_{\mathbf{p}}}{d\lambda} \in L^q$ if and only if $1 \leq q < \frac{r_{\max}}{r_{\max}-1}$ and in the case that $r_{\max} = 1$ we have $\frac{d\mu_{\mathbf{p}}}{d\lambda} \in L^q$ for all $q \in [1, \infty]$. In view of this, one could wonder for which $q > 1$ we have $\frac{d\mu_{\mathbf{p}}}{d\lambda} \in L^q$ in the intermediate case that $\ell_{\max} = 1$ and $\sum_{b \in \Sigma_B^1} \frac{p_b}{|DT_b(c)|} \geq 1$, i.e. if c is not superattracting for any bad map and the bad maps are not expanding on average at c.

References

Aar97 [1] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

AbbGhaHom [2] N. Abbasi, M. Gharaei, and A. J. Homburg. Iterated function systems of logistic maps: synchronization and intermittency. *Nonlinearity*, 31(8):3880–3913, 2018.

ALP [3] J. F. Alves, S. Luzzatto, and V. Pinheiro. Lyapunov exponents and rates of mixing for one-dimensional maps. *Ergodic Theory Dynam. Systems*, 24(3):637–657, 2004.

AD00 [4] K. B. Athreya and J. Dai. Random logistic maps. I. *J. Theoret. Probab.*, 13(2):595–608, 2000.

AthSch [5] K. B. Athreya and H.-J. Schuh. Random logistic maps. II. The critical case. *J. Theoret. Probab.*, 16(4):813–830 (2004), 2003.

BB16 [6] W. Bahsoun and C. Bose. Mixing rates and limit theorems for random intermittent maps. *Nonlinearity*, 29(4):1417–1433, 2016.

BBD14 [7] W. Bahsoun, C. Bose, and Y. Duan. Decay of correlation for random intermittent maps. *Nonlinearity*, 27(7):1543–1554, 2014.

BBR19 [8] W. Bahsoun, C. Bose, and M. Ruziboev. Quenched decay of correlations for slowly mixing systems. *Trans. Amer. Math. Soc.*, 372(9):6547–6587, 2019.

BRS20 [9] W. Bahsoun, M. Ruziboev, and B. Saussol. Linear response for random dynamical systems. *Adv. Math.*, 364:107011, 44, 2020.

BS16 [10] W. Bahsoun and B. Saussol. Linear response in the intermittent family: differentiation in a weighted C^0-norm. *Discrete Contin. Dyn. Syst.*, 36(12):6657–6668, 2016.

BT16 [11] V. Baladi and M. Todd. Linear response for intermittent maps. *Comm. Math. Phys.*, 347(3):857–874, 2016.

BPV [12] P. Bergé, Y. Pomeau, and C. Vidal. *Order within chaos.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York; Hermann, Paris, 1986. Towards a deterministic approach to turbulence, With a preface by David Ruelle, Translated from the French by Laurette Tuckerman.

BLS [13] H. Bruin, S. Luzzatto, and S. Van Strien. Decay of correlations in one-dimensional dynamics. *Ann. Sci. École Norm. Sup. (4)*, 36(4):621–646, 2003.

CF90 [14] P. Collet and P. Ferrero. Some limit ratio theorem related to a real endomorphism in case of a neutral fixed point. *Ann. Inst. H. Poincaré Phys. Théor.*, 52(3):283–301, 1990.

dMvS93 [15] W. de Melo and S. van Strien. *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.

FFTV [16] A. C. M. Freitas, J. M. Freitas, M. Todd, and S. Vaienti. Rare events for the Manneville-Pomeau map. *Stochastic Process. Appl.*, 126(11):3463–3479, 2016.

Fro99 [17] G. Froyland. Ulam's method for random interval maps. *Nonlinearity*, 12(4):1029–1052, 1999.

GO04 [18] S. Gouëzel. Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields*, 128(1):82–122, 2004.

GO07 [19] S. Gouëzel. A Borel-Cantelli lemma for intermittent interval maps. *Nonlinearity*, 20(6):1491–1497, 2007.

Gou07 [20] S. Gouëzel. Statistical properties of a skew product with a curve of neutral points. *Ergodic Theory Dynam. Systems*, 27(1):123–151, 2007.

HP [21] A. J. Homburg and H. Peters. Critical intermittency in rational maps. `https://staff.fnwi.uva.nl/a.j.homburg/Files/critical-ajh.pdf`.

Hu01 [22] H. Hu. Statistical properties of some almost hyperbolic systems. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 367–384. Amer. Math. Soc., Providence, RI, 2001.

KKV17 [23] C. Kalle, T. Kempton, and E. Verbitskiy. The random continued fraction transformation. *Nonlinearity*, 30(3):1182–1203, 2017.

Las04 [24] A. Lasota, J. Myjak, and T. Szarek. Markov operators and semifractals. In *Fractal geometry and stochastics III*, volume 57 of *Progr. Probab.*, pages 3–22. Birkhäuser, Basel, 2004.

LSV [25] C. Liverani, B. Saussol, and S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685, 1999.

ManPum2 [26] P. Manneville and Y. Pomeau. Different ways to turbulence in dissipative dynamical systems. *Phys. D*, 1(2):219–226, 1980.

Mor85 [27] T. Morita. Asymptotic behavior of one-dimensional random dynamical systems. *J. Math. Soc. Japan*, 37(4):651–663, 1985.

NowvSt [28] T. Nowicki and S. van Strien. Invariant measures exist under a summability condition for unimodal maps. *Invent. Math.*, 105(1):123–136, 1991.

PW99 [29] M. Pollicott and H. Weiss. Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation. *Comm. Math. Phys.*, 207(1):145–171, 1999.

PY01 [30] M. Pollicott and M. Yuri. Statistical properties of maps with indifferent periodic points. *Comm. Math. Phys.*, 217(3):503–520, 2001.

ManPum [31] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.*, 74(2):189–197, 1980.

Tha80 [32] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.

Tol03 [33] R. Toledano. A note on the Lebesgue differentiation theorem in spaces of homogeneous type. *Real Anal. Exchange*, 29(1):335–339, 2003/04.

Y99 [34] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188, 1999.

Zwe03 [35] R. Zweimüller. Stable limits for probability preserving maps with indifferent fixed points. *Stoch. Dyn.*, 3(1):83–99, 2003.