Series of convex functions: subdifferential, conjugate and applications to entropy minimization

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Consider f, f_n proper convex functions defined on a Banach space X such that

$$f(x) = \sum_{n\geq 1} f_n(x) \quad (x \in X).$$

X. Y. Zheng (1998) showed that the subdifferential $\partial f(x)$ of f at x is given by the formula

$$\partial f(x) = w^*\text{-}\sum_{n\geq 1} \partial f_n(x) \tag{1}$$

for all $x \in \operatorname{int}(\operatorname{dom} f)$ whenever f and f_n are continuous on $\operatorname{int}(\operatorname{dom} f)$

Our aim is to provide a proof for (1) in locally convex spaces under the same conditions on f, f_n and x, and to show that it can be used to obtain the formula

$$f^*(x^*) = \min\left\{ \sum_{n\geq 1} f_n^*(x_n^*) \mid x_n^* \in \operatorname{dom} f_n^* \ \forall n \geq 1,\ x^* = w^*\text{-}\sum_{n\geq 1} x_n^* \right\} \tag{2}$$

for all $x^* \in \partial f(\operatorname{int}\operatorname{dom} f)$.

By examples in finite dimensional spaces we show that the above conditions for the validity of formulas (1) and (2) are essential.

We then show how the previous results can be used to obtain rigorously the maximum of the Boltzmann entropy for a classical choice of level energies $(e_i)_{i\in I}$, as well as the corresponding Boltzmann distribution.

References

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